
Structure of Cesàro function spaces [☆]**by Sergei V. Astashkin** ^a and **Lech Maligranda** ^b^a *Department of Mathematics and Mechanics, Samara State University, Acad. Pavlova 1, 443011 Samara, Russia*^b *Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden*

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ABSTRACT

The structure of the Cesàro function spaces Ces_p on both $[0, 1]$ and $[0, \infty)$ for $1 < p \leq \infty$ is investigated. We find their dual spaces, which equivalent norms have different description on $[0, 1]$ and $[0, \infty)$. The spaces Ces_p for $1 < p < \infty$ are not reflexive but strictly convex. They are not isomorphic to any L^q space with $1 \leq q \leq \infty$. They have “near zero” complemented subspaces isomorphic to l^p and “in the middle” contain an asymptotically isometric copy of l^1 and also a copy of $L^1[0, 1]$. They do not have Dunford–Pettis property but they do have the weak Banach–Saks property. Cesàro function spaces on $[0, 1]$ and $[0, \infty)$ are isomorphic for $1 < p \leq \infty$. Moreover, we give characterizations in terms of p and q when $\text{Ces}_p[0, 1]$ contains an isomorphic copy of l^q .

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Let $1 \leq p \leq \infty$. The *Cesàro sequence space* ces_p is defined as the set of all real sequences $x = \{x_k\}$ such that

$$\|x\|_{c(p)} = \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n |x_k| \right)^p \right]^{1/p} < \infty \quad \text{when } 1 \leq p < \infty$$

and

$$\|x\|_{c(\infty)} = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^n |x_k| < \infty \quad \text{when } p = \infty.$$

The *Cesàro function spaces* $\text{Ces}_p = \text{Ces}_p(I)$ are the classes of Lebesgue measurable real functions f on $I = [0, 1]$ or $I = [0, \infty)$ such that

$$\|f\|_{C(p)} = \left[\int_I \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \right]^{1/p} < \infty \quad \text{for } 1 \leq p < \infty$$

and

$$\|f\|_{C(\infty)} = \sup_{x \in I, x > 0} \frac{1}{x} \int_0^x |f(t)| dt < \infty \quad \text{for } p = \infty.$$

The Cesàro sequence spaces ces_p and ces_∞ appeared in 1968 in connection with the problem of the Dutch Mathematical Society to find their duals. Some investigations of ces_p were done by Shiue [50] in 1970. Then Leibowitz [36] and Jagers [26] proved that $\text{ces}_1 = \{0\}$, ces_p are separable reflexive Banach spaces for $1 < p < \infty$ and the l^p spaces are continuously and strictly embedded into ces_p for $1 < p \leq \infty$. More precisely, $\|x\|_{c(p)} \leq p' \|x\|_p$ for all $x \in l^p$ with $p' = \frac{p}{p-1}$ when $1 < p < \infty$ and $p' = 1$ when $p = \infty$. Moreover, if $1 < p < q \leq \infty$, then $\text{ces}_p \subset \text{ces}_q$ with continuous strict embedding. Bennett [8] proved that ces_p for $1 < p < \infty$ are not isomorphic to any l^q space with $1 \leq q \leq \infty$ (see also [45] for another proof).

Several geometric properties of the Cesàro sequence spaces ces_p were studied in the last years by many mathematicians (see e.g. [10–16, 34]). Some more results on ces_p can be found in two books [8, 39].

In 1999–2000 it was proved by Cui and Hudzik [11], Cui, Hudzik and Li [14] and Cui, Meng and Pluciennik [16] that the Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property (cf. also [10, Part 9]). Maligranda, Petrot and Sountai [45] proved that the Cesàro sequence spaces ces_p for $1 < p < \infty$ are not uniformly non-square, that is, there are sequences $\{x_n\}$ and $\{y_n\}$ on the unit sphere such that $\lim_{n \rightarrow \infty} \min(\|x_n + y_n\|_{c(p)}, \|x_n - y_n\|_{c(p)}) = 2$. They even proved that these spaces are not B -convex.

The Cesàro function spaces $\text{Ces}_p[0, \infty)$ for $1 \leq p \leq \infty$ were considered by Shiue [51], Hassard and Hussein [25] and Sy, Zhang and Lee [54]. The space $\text{Ces}_\infty[0, 1]$

appeared already in 1948 and it is known as the Korenblyum, Krein and Levin space K (see [31] and [59]).

Recently, we proved in the paper [4] that, in contrast to Cesàro sequence spaces, the Cesàro function spaces $\text{Ces}_p(I)$ on both $I = [0, 1]$ and $I = [0, \infty)$ for $1 < p < \infty$ are not reflexive and they do not have the fixed point property. In other paper [5] we investigated Rademacher sums in $\text{Ces}_p[0, 1]$ for $1 \leq p \leq \infty$. The description is different for $1 \leq p < \infty$ and $p = \infty$.

We recall some notions and definitions which we will need later on. By $L^0 = L^0(I)$ we denote the set of all equivalence classes of real-valued Lebesgue measurable functions defined on $I = [0, 1]$ or $I = [0, \infty)$. A *normed function lattice* or *normed ideal space* $X = (X, \|\cdot\|)$ (on I) is understood to be a normed space in $L^0(I)$, which satisfies the so-called ideal property: if $|f| \leq |g|$ a.e. on I and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$. If, in addition, X is a complete space, then we say that X is a *Banach function lattice* or a *Banach ideal space* (on I). Sometimes we write $\|\cdot\|_X$ to be sure in which space the norm is taken.

For two normed ideal spaces X and Y on I the symbol $X \hookrightarrow Y$ means that $X \subset Y$ and the imbedding is continuous, and the symbol $X \xhookrightarrow{C} Y$ means that $X \hookrightarrow Y$ with the inequality $\|x\|_Y \leq C\|x\|_X$ for all $x \in X$. Moreover, notation $X \simeq Y$ means that these two spaces are isomorphic.

For a normed ideal space $X = (X, \|\cdot\|)$ on I and $1 \leq p < \infty$ the *p-convexification* $X^{(p)}$ of X is the space of all $f \in L^0(I)$ such that $|f|^p \in X$ with the norm

$$\|f\|_{X^{(p)}} := \||f|^p\|_X^{1/p}.$$

$X^{(p)}$ is also a normed ideal space on I .

For a normed ideal space $X = (X, \|\cdot\|)$ on I the *Köthe dual* (or *associated space*) X' is the space of all $f \in L^0(I)$ such that the *associate norm*

$$\|f\|' := \sup_{g \in X, \|g\|_X \leq 1} \int_I |f(x)g(x)| dx$$

is finite. The Köthe dual $X' = (X', \|\cdot\|')$ is a Banach ideal space. Moreover, $X \subset X''$ with $\|f\| \leq \|f\|''$ for all $f \in X$ and we have equality $X = X''$ with $\|f\| = \|f\|''$ if and only if the norm in X has the *Fatou property*, that is, if $0 \leq f_n \nearrow f$ a.e. on I and $\sup_{n \in \mathbb{N}} \|f_n\| < \infty$, then $f \in X$ and $\|f_n\| \nearrow \|f\|$.

For a normed ideal space $X = (X, \|\cdot\|)$ on I with the Köthe dual X' we have the following Hölder type inequality: if $f \in X$ and $g \in X'$, then fg is integrable and

$$\int_I |f(x)g(x)| dx \leq \|f\|_X \|g\|_{X'}.$$

A function f in a normed ideal space X on I is said to have *absolutely continuous norm* in X if, for any decreasing sequence of Lebesgue measurable sets $A_n \subset I$ with

empty intersection, we have that $\|f \chi_{A_n}\| \rightarrow 0$ as $n \rightarrow \infty$. The set of all functions in X with absolutely continuous norm is denoted by X_a . If $X_a = X$, then the space X itself is said to have *absolutely continuous norm*. For a normed ideal space X with absolutely continuous norm, the Köthe dual X' and the dual space X^* coincide. Moreover, a Banach ideal space X is reflexive if and only if both X and its associate space X' have absolutely continuous norms.

For general properties of normed and Banach ideal spaces we refer to the books Krein, Petunin and Semenov [32], Kantorovich and Akilov [28], Bennett and Sharpley [9], Lindenstrauss and Tzafriri [38] and Maligranda [43].

The paper is organized as follows: In Section 1 some necessary definitions and notation are collected. In Section 2 some simple results on Cesàro function spaces are presented. In particular, we can see that the Cesàro function spaces $\text{Ces}_p(I)$ are not reflexive but strictly convex for all $1 < p < \infty$.

Sections 3 and 4 contain results on the dual and Köthe dual of Cesàro function spaces. There is a big difference between the cases on $[0, \infty)$ and on $[0, 1]$, as we can see from Theorems 2 and 3. This was also the reason why we put these investigations into two parts. Important in our investigations were earlier results on the Köthe dual $(\text{ces}_p)'$ and remark on the Köthe dual $(\text{Ces}_p[0, \infty))'$ due to Bennett [8]. This remark was recently proved, even for more general spaces, by Kerman, Milman and Sinnamon [30]. Luxemburg and Zaanen [42] gave a description of the Köthe dual $(\text{Ces}_\infty[0, 1])'$.

Section 5 deals with the p -concavity and cotype of Cesàro sequence spaces ces_p and Cesàro function spaces $\text{Ces}_p(I)$. It is shown, in Theorem 4, that they are p -concave for $1 < p < \infty$ with constant one and, thus, they have cotype $\max(p, 2)$.

In Section 6 it is proved, in Theorem 6, that the Cesàro function spaces $\text{Ces}_p(I)$ contain an order isomorphic and complemented copy of l^p . Therefore, they do not have the Dunford–Pettis property. This result and cotype property imply that $\text{Ces}_p(I)$ are not isomorphic to any $L^q(I)$ space for $1 \leq q \leq \infty$ (Theorem 7).

The authors proved in [4] that “in the middle” Cesàro function spaces $\text{Ces}_p(I)$ contain an asymptotically isometric copy of l^1 and consequently they are not reflexive and do not have the fixed point property. This is a big difference with Cesàro sequence spaces ces_p , which for $1 < p < \infty$ are reflexive and which have the fixed point property.

Section 7 contains the proof that the Cesàro function spaces $\text{Ces}_p[0, 1]$ for $1 \leq p < \infty$ have the weak Banach–Saks property. Important role in the proof will be played by the description of the dual space given in Section 4.

In Section 8 we present a construction showing that the Cesàro function spaces $\text{Ces}_p[0, \infty)$ and $\text{Ces}_p[0, 1]$ for $1 < p \leq \infty$ are isomorphic. The isomorphisms are different in the cases $1 < p < \infty$ and $p = \infty$.

In Section 9 it is proved that $\text{Ces}_p[0, 1]$ contains an isomorphic copy of l^q if and only if $q \in [1, 2]$ for the case $1 \leq p \leq 2$ and in the case when $p > 2$ this can happen when either $q \in [1, 2]$ or $q = p$. This result is, in fact, different from the one for $L^p[0, 1]$ space.

The Cesàro function spaces $\text{Ces}_p[0, \infty)$ for $1 \leq p \leq \infty$ were considered by Shiue [51], Hassard and Hussein [25] and Sy, Zhang and Lee [54]. The space $\text{Ces}_\infty[0, 1]$ appeared in 1948 and it is known as the Korenblyum, Krein and Levin space K (see [31] and [59, p. 26 and 61]).

We collect some known or clear properties of $\text{Ces}_p(I)$ for both $I = [0, 1]$ and $I = [0, \infty)$ in one place.

Theorem 1.

- (a) If $1 < p \leq \infty$, then $\text{Ces}_p(I)$ are Banach spaces, $\text{Ces}_1[0, 1] = L_w^1$ with the weight $w(t) = \ln \frac{1}{t}$, $t \in (0, 1]$ and $\text{Ces}_1[0, \infty) = \{0\}$.
- (b) The spaces $\text{Ces}_p(I)$ are separable for $1 < p < \infty$ and $\text{Ces}_\infty(I)$ is non-separable.
- (c) If $1 < p \leq \infty$, then $L^p(I) \xhookrightarrow{p'} \text{Ces}_p(I)$, where $p' = \frac{p}{p-1}$ and the embedding is strict.
- (d) If $1 < p < \infty$, then $\text{Ces}_p[0, 1]_{|[0,a]} \hookrightarrow L^1[0, a]$ for any $a \in (0, 1)$ but not for $a = 1$ and $\text{Ces}_p[0, \infty)_{|[0,a]} \hookrightarrow L^1[0, a]$ for any $0 < a < \infty$ but not for $a = \infty$, that is, $\text{Ces}_p[0, \infty) \not\subset L^1[0, \infty)$. Moreover, $\text{Ces}_\infty[0, 1] \xhookrightarrow{1} L^1[0, 1]$.
- (e) If $1 < p < q \leq \infty$, then $\text{Ces}_q[0, 1] \xhookrightarrow{1} \text{Ces}_p[0, 1]$ and the embedding is strict.
- (f) The spaces $\text{Ces}_p(I)$ are not rearrangement invariant.
- (g) The spaces $\text{Ces}_p(I)$ are not reflexive.
- (h) The spaces $\text{Ces}_p(I)$ for $1 < p < \infty$ are strictly convex, that is, if $\|f\|_{C(p)} = \|g\|_{C(p)} = 1$ and $f \neq g$, then $\|\frac{f+g}{2}\|_{C(p)} < 1$.

Proof. (a), (b) Shiue [51] and Hassard and Hussein [25] proved that $\text{Ces}_p(I)$ are separable Banach spaces for $1 < p < \infty$ and non-separable ones for $p = \infty$. We only show here that $\text{Ces}_1[0, 1]$ is a weighted $L_w^1[0, 1]$ space with the weight $w(t) = \ln \frac{1}{t}$ for $0 < t \leq 1$ and $\text{Ces}_1[0, \infty) = \{0\}$. In fact,

$$(1) \quad \int_0^1 \left(\frac{1}{x} \int_0^x |f(t)| dt \right) dx = \int_0^1 \left(\int_t^1 \frac{1}{x} dx \right) |f(t)| dt = \int_0^1 |f(t)| \ln \frac{1}{t} dt.$$

Moreover, if $f \in L^0[0, \infty)$ and $|f(x)| > 0$ for $x \in A$ with $0 < m(A) < \infty$, then there exists sufficiently large $a > 0$ such that $\delta = \int_0^a |f(t)| dt > 0$. Therefore, for $b > a$, it yields that

$$\begin{aligned} \int_0^b \left(\frac{1}{x} \int_0^x |f(t)| dt \right) dx &\geq \int_a^b \left(\frac{1}{x} \int_0^x |f(t)| dt \right) dx \\ &\geq \int_a^b \left(\frac{1}{x} \int_0^a |f(t)| dt \right) dx \end{aligned}$$

$$= \delta \ln \frac{b}{a} \rightarrow \infty \quad \text{as } b \rightarrow \infty.$$

Thus $f \notin \text{Ces}_1[0, \infty)$.

(c) Considering the Hardy operator $Hf(x) = \frac{1}{x} \int_0^x f(t) dt$ and using the Hardy inequality (cf. [24, Theorem 327] and [33, Theorem 2]) we obtain that

$$\|f\|_{C(p)} = \|H(|f|)\|_p \leq p' \|f\|_p$$

for all $f \in L^p(I)$, which means that the $L^p(I) \xrightarrow{p'} \text{Ces}_p(I)$ for $1 < p \leq \infty$.

The embeddings are strict. For example, $f = \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \chi_{[n^2-1, n^2)} \in \text{Ces}_p(I) \setminus L^p(I)$ for $I = [0, \infty)$ and $1 < p < \infty$.

(d) If $0 < a < 1$ and $\text{supp } f \subset [0, a]$, then

$$\begin{aligned} \|f\|_{C(p)} &\geq \left(\int_a^1 \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \right)^{1/p} \\ &\geq \left(\int_a^1 \left(\frac{1}{x} \int_0^a |f(t)| dt \right)^p dx \right)^{1/p} = \int_0^a |f(t)| dt \left(\frac{1-a^{1-p}}{p-1} \right)^{1/p}. \end{aligned}$$

For $a = 1$ this is not the case. In fact, consider function $f(x) = \frac{1}{1-x}$ for $x \in [0, 1)$. Then $\frac{1}{x} \int_0^x f(t) dt = \frac{1}{x} \ln \frac{1}{1-x}$ and

$$\begin{aligned} \|f\|_{C(p)}^p &= \int_0^1 \left(\frac{1}{x} \ln \frac{1}{1-x} \right)^p dx = \int_1^\infty \left(\frac{t \ln t}{t-1} \right)^p \frac{dt}{t^2} \\ &\leq c + \int_2^\infty \frac{(2 \ln t)^p}{t^2} dt < \infty \end{aligned}$$

and, hence, $f \in \text{Ces}_p[0, 1]$ for any $1 \leq p < \infty$ but clearly, $f \notin L^1[0, 1]$.

In the case of $\text{Ces}_p[0, \infty)$ we will have for $0 < a < \infty$ with $\text{supp } f \subset [0, a]$ and $p \in (1, \infty)$,

$$\begin{aligned} \|f\|_{C(p)} &\geq \left(\int_a^\infty \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \right)^{1/p} \\ &\geq \left(\int_a^\infty \left(\frac{1}{x} \int_0^a |f(t)| dt \right)^p dx \right)^{1/p} = \int_0^a |f(t)| dt \frac{1}{(p-1)a^{1-1/p}}. \end{aligned}$$

For the function $f(x) = \frac{1}{x} \chi_{[1, \infty)}(x)$, $x \in (0, \infty)$ we have $\frac{1}{x} \int_0^x f(t) dt = \frac{1}{x} \ln x$ ($x \geq 1$) and

$$\|f\|_{C(p)}^p = \int_1^\infty \left(\frac{\ln x}{x}\right)^p dx < \infty.$$

Thus, $f \in \text{Ces}_p[0, \infty)$ for any $1 < p < \infty$, but clearly $f \notin L^1[0, \infty)$.

(e) If $1 < p < q \leq \infty$, then $L^q[0, 1] \xrightarrow{1} L^p[0, 1]$ and the embedding is strict, and, thus,

$$\|f\|_{C(p)} = \|H(|f|)\|_p \leq \|H(|f|)\|_q = \|f\|_{C(q)}$$

for all $f \in \text{Ces}_q[0, 1]$, that is, $\text{Ces}_q[0, 1] \xrightarrow{1} \text{Ces}_p[0, 1]$ and the embedding is strict since for positive decreasing functions the norms of Ces_p and L^p are equivalent. The last statement follows from the fact that for a positive decreasing function f on I we have $f(x) \leq \frac{1}{x} \int_0^x f(t) dt$ for $x \in I$ and so

$$\|f\|_p \leq \|Hf\|_p = \|f\|_{C(p)} \leq p' \|f\|_p \quad \text{for any } 0 \leq f \in L^p(I).$$

(f) Consider $f(x) = \frac{1}{1-x}$ for $x \in [0, 1)$. Then, as it was shown in (d), $f \in \text{Ces}_p[0, 1]$ for any $1 \leq p < \infty$. However, its non-increasing rearrangement $f^*(t) = t^{-1}$ ($0 < t \leq 1$) does not belong to $\text{Ces}_p[0, 1]$ for any $1 \leq p \leq \infty$ and therefore the space $\text{Ces}_p[0, 1]$ is not rearrangement invariant for $1 \leq p < \infty$. In the case when $p = \infty$ we can take the function $g(x) = \frac{1}{\sqrt{1-x}}$, $x \in [0, 1)$ for which $\frac{1}{x} \int_0^x g(t) dt = \frac{2}{x}(1 - \sqrt{1-x}) = \frac{2}{1+\sqrt{1-x}}$ and so $\|g\|_{C(\infty)} = 2$ and for its rearrangement $g^*(t) = t^{-1/2}$, $t \in (0, 1)$ we have $\|g^*\|_{C(\infty)} = \sup_{t \in (0, 1)} 2t^{-1/2} = \infty$, that is, $g^* \notin \text{Ces}_\infty[0, 1]$ and the space $\text{Ces}_\infty[0, 1]$ is not rearrangement invariant. Similarly, we can consider the case when $I = [0, \infty)$.

(g) If $1 < p < \infty$, then $\text{Ces}_p(I)$ contains a copy of $L^1(I)$ (cf. [4], Lemma 1 for $I = [0, 1]$ and Theorem 2 for $I = [0, \infty)$) and therefore, in particular, these spaces cannot be reflexive. Of course, $\text{Ces}_1[0, 1] = L^1(\ln 1/t)$ is not reflexive and the space $\text{Ces}_\infty(I)$ does not have absolutely continuous norm and therefore is also not reflexive.

(h) Assume that $\|f\|_{C(p)} = \|g\|_{C(p)} = 1$ and $\|f+g\|_{C(p)} = 2$; then $\|H(|f|)\|_{L^p} = \|H(|g|)\|_{L^p} = 1$ and

$$\begin{aligned} 2 &= \|f+g\|_{C(p)} = \|H(|f+g|)\|_{L^p} \\ &\leq \|H(|f|) + H(|g|)\|_{L^p} \leq \|H(|f|)\|_{L^p} + \|H(|g|)\|_{L^p} \\ &= \|f\|_{C(p)} + \|g\|_{C(p)} = 2. \end{aligned}$$

Thus $\|H(|f|) + H(|g|)\|_{L^p} = 2$ and by the strict convexity of $L^p(I)$ for $1 < p < \infty$ and the above estimates we obtain that $H(|f|)(x) = H(|g|)(x)$ for almost all x in I . Therefore, $|f(x)| = |g(x)|$ for almost all $x \in I$. We want to show that this implies that $f(x) = g(x)$ for almost all $x \in I$. Assume on the contrary that $f \neq g$ on I , that

is, there exists a set $A \subset I$ of positive measure $m(A) > 0$ such that $f(x) \neq g(x)$ for all $x \in A$. Then $f(x) = -g(x)$ and $|f(x)| > 0$ for $x \in A$. Moreover, if $B = \{x \in I: m([0, x] \cap (I \setminus A)) < x\}$, then $m(B) > 0$ and

$$\int_0^x \left| \frac{f(t) + g(t)}{2} \right| dt = \int_{[0, x] \cap (I \setminus A)} |f(t)| dt < \int_0^x |f(t)| dt$$

for all $x \in B$. Therefore,

$$\begin{aligned} 1 &= \left\| \frac{f+g}{2} \right\|_{C(p)}^p = \int_I \left(\frac{1}{x} \int_0^x \left| \frac{f(t) + g(t)}{2} \right| dt \right)^p dx \\ &< \int_I \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx = \|f\|_{C(p)}^p = 1, \end{aligned}$$

which is a contradiction and the proof is complete. \square

3. THE DUAL SPACES OF THE CESÀRO FUNCTION SPACES $\text{Ces}_p[0, \infty)$

We describe the dual and Köthe dual spaces of $\text{Ces}_p(I)$ for $1 < p < \infty$ in the case $I = [0, \infty)$. The description appeared as remark in Bennett [8] paper but it was proved recently, even for more general spaces, by Kerman, Milman and Sinnamon [30, Theorem D] and they used in the proof some of Sinnamon results [53, Theorem 2.1] and [52, Proposition 2.1 and Lemma 3.2].

We present here another proof following the Bennett's idea for Cesàro sequence spaces together with factorization theorems which are of independent interest. Since the case $I = [0, 1]$ is essentially different it will be considered in the next section.

Theorem 2. *Let $I = [0, \infty)$. If $1 < p < \infty$, then*

$$(2) \quad (\text{Ces}_p)^* = (\text{Ces}_p)' = D(p'), \quad p' = \frac{p}{p-1},$$

with $\|f\|_{C(p)'} \leq p' \|f\|_{D(p')} \leq 8p' \|f\|_{C(p)'}$, where the norm in $D(p')$ is given by formula

$$(3) \quad \|f\|_{D(p')} = \|\tilde{f}\|_{L^{p'}} \quad \text{with} \quad \tilde{f}(x) = \text{ess} \sup_{t \in [x, \infty)} |f(t)|.$$

We need the definition of the $G(p)$ space for $1 \leq p < \infty$, which is the p -convexification of $\text{Ces}_\infty[0, \infty)$, that is, its norm is given by the functional

$$\|f\|_{G(p)} = \| |f|^p \|_{C(\infty)}^{1/p} = \sup_{x>0} \left(\frac{1}{x} \int_0^x |f(t)|^p dt \right)^{1/p}.$$

Proposition 1. *If $1 < p < \infty$, then*

$$(4) \quad \text{Ces}_p = L^p \cdot G(p'),$$

that is, $f \in \text{Ces}_p$ if and only if $f = gh$ with $g \in L^p, h \in G(p')$ and

$$(5) \quad \|f\|_{C(p)} \approx \inf \|g\|_p \|h\|_{G(p')},$$

where infimum is taken over all factorizations $f = gh$ with $g \in L^p, h \in G(p')$.

Proof. “Imbedding \hookrightarrow ”. For $f \in \text{Ces}_p, f \not\equiv 0$ let

$$k(x) = \int_x^\infty u^{-p} \left(\int_0^u |f(t)| dt \right)^{p-1} du, \quad x > 0.$$

Then $k(x) > 0, k$ is decreasing and by the Hölder–Rogers inequality

$$\begin{aligned} k(x) &= \int_x^\infty u^{-1} \left(\frac{1}{u} \int_0^u |f(t)| dt \right)^{p-1} du \\ &\leq \left(\int_x^\infty u^{-p} du \right)^{1/p} \left(\int_x^\infty \left(\frac{1}{u} \int_0^u |f(t)| dt \right)^p du \right)^{1/p'} \\ &= \frac{1}{(p-1)^{1/p} x^{1-1/p}} \|f\|_{C(p)}^{p-1}. \end{aligned}$$

We consider the factorization $f = g \cdot h$, where

$$g(x) = (|f(x)|k(x))^{1/p} \operatorname{sgn} f(x) \quad \text{and} \quad h(x) = |f(x)|^{1/p'} k(x)^{-1/p}.$$

Then

$$\begin{aligned} \|g\|_p^p &= \int_0^\infty |f(x)| \int_x^\infty u^{-p} \left(\int_0^u |f(t)| dt \right)^{p-1} du dx \\ &= \int_0^\infty u^{-p} \left(\int_0^u |f(t)| dt \right)^{p-1} \int_0^u |f(x)| dx du = \|f\|_{C(p)}^p \end{aligned}$$

and, by the Hölder–Rogers inequality,

$$\begin{aligned} \left(\int_0^x |h(t)|^{p'} dt \right)^p &= \left(\int_0^x |f(t)|^{1/p'} |f(t)|^{1/p} k(t)^{-p'/p} dt \right)^p \\ &\leq \left(\int_0^x |f(t)| dt \right)^{p-1} \left(\int_0^x |f(t)| k(t)^{-p'} dt \right). \end{aligned}$$

Hence, by the above and using the fact that k is decreasing, it yields that

$$\begin{aligned}
& \int_x^\infty \left(s^{-1} \int_0^x |h(t)|^{p'} dt \right)^p ds \\
& \leq \int_x^\infty s^{-p} \left[\left(\int_0^x |f(t)| dt \right)^{p-1} \int_0^x |f(t)| k(t)^{-p'} dt \right] ds \\
& = k(x) \int_0^x |f(t)| k(t)^{-p'} dt \\
& \leq \int_0^x |f(t)| k(t)^{1-p'} dt = \int_0^x |h(t)|^{p'} dt
\end{aligned}$$

or, equivalently,

$$\int_x^\infty s^{-p} ds \left(\int_0^x |h(t)|^{p'} dt \right)^{p-1} \leq 1,$$

which means that

$$\left(\int_0^x |h(t)|^{p'} dt \right)^{p-1} \leq (p-1)x^{p-1}$$

and, hence,

$$\sup_{x>0} \frac{1}{x} \int_0^x |h(t)|^{p'} dt \leq (p-1)^{1/(p-1)}$$

or $\|h\|_{G(p')} \leq (p-1)^{1/p}$. We have proved that

$$\text{Ces}_p \subset L^p \cdot G(p')$$

and

$$\inf\{\|g\|_{L^p} \|h\|_{G(p')}: f = g \cdot h\} \leq (p-1)^{1/p} \|f\|_{C(p)}.$$

“*Imbedding* \leftarrow ”. Let $f = g \cdot h$ with $g \in L^p$ and $h \in G(p')$. Then

$$\int_0^x |h(t)|^{p'} dt \leq \|h\|_{G(p')}^{p'} \int_0^x dt$$

and then, for any positive decreasing function w on $(0, \infty)$, we have by [32, property 18⁰, p. 72] that

$$\int_0^x |h(t)|^{p'} w(t) dt \leq \|h\|_{G(p')}^{p'} \int_0^x w(t) dt.$$

By the Hölder–Rogers inequality we find that

$$\begin{aligned} \left(\int_0^x |f(t)| dt \right)^p &= \left(\int_0^x |g(t)| w(t)^{-1/p'} |h(t)| w(t)^{1/p'} dt \right)^p \\ &\leq \int_0^x |g(t)|^p w(t)^{1-p} dt \left(\int_0^x |h(t)|^{p'} w(t) dt \right)^{p-1} \\ &\leq \int_0^x |g(t)|^p w(t)^{1-p} dt \|h\|_{G(p')}^p \left(\int_0^x w(t) dt \right)^{p-1} \end{aligned}$$

and, thus,

$$\begin{aligned} &\int_0^\infty \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \\ &\leq \int_0^\infty x^{-p} \left(\int_0^x |g(t)|^p w(t)^{1-p} dt \right) \left(\int_0^x w(t) dt \right)^{p-1} dx \|h\|_{G(p')}^p. \end{aligned}$$

Taking in the last estimate $w(t) = t^{-1/p}$ we obtain that

$$\begin{aligned} \|f\|_{C(p)}^p &\leq \int_0^\infty x^{-p} \left(\int_0^x |g(t)|^p t^{1-1/p} dt \right) \left(\frac{x^{1-1/p}}{1-1/p} \right)^{p-1} dx \|h\|_{G(p')}^p \\ &= (p')^{p-1} \int_0^\infty \left(\int_0^x |g(t)|^p t^{1-1/p} dt \right) x^{1/p-2} dx \|h\|_{G(p')}^p \\ &= (p')^{p-1} \int_0^\infty \left(\int_t^\infty x^{1/p-2} dx \right) |g(t)|^p t^{1-1/p} dt \|h\|_{G(p')}^p \\ &= (p')^p \int_0^\infty |g(t)|^p dt \|h\|_{G(p')}^p = (p')^p \|g\|_p^p \|h\|_{G(p')}^p \end{aligned}$$

or

$$\|f\|_{C(p)} \leq p' \|g\|_p \|h\|_{G(p')},$$

that is, $L^p \cdot G(p') \subset \text{Ces}_p$ and

$$\|f\|_{C(p)} \leq p' \inf\{\|g\|_p \|h\|_{G(p')}: f = gh\}.$$

Putting these facts together we have that $\text{Ces}_p \xrightarrow{(p-1)^{1/p}} L^p \cdot G(p') \xrightarrow{p'} \text{Ces}_p$ and the proof of Proposition 1 is complete. \square

Proposition 2. *If $1 \leq p < \infty$, then*

$$D(p) \cdot G(p) = L^p$$

and

$$\|f\|_{L^p} = \inf\{\|g\|_{D(p)} \|h\|_{G(p)}: f = gh, g \in D(p), h \in G(p)\}.$$

Moreover, $G(1)' = D(1)$ with equality of the norms.

Proof. It suffices to prove the statement for $p = 1$ because the general result follows by p -convexification. Suppose that $f = gh$ with $g \in D(1)$, $h \in G(1)$. Then

$$\|f\|_{L^1} = \int_0^\infty |g(t)h(t)| dt \leq \int_0^\infty \tilde{g}(t)|h(t)| dt.$$

Moreover, from the definition of the norm in $G(1)$ it follows that

$$\int_0^t |h(s)| ds \leq \|h\|_{G(1)} t = \|h\|_{G(1)} \int_0^t \chi_{[0,\infty)}(s) ds, \quad t > 0.$$

Therefore, since \tilde{g} decreases it follows by [32, property 18⁰, p. 72], we find that

$$\|f\|_{L^1} \leq \|h\|_{G(1)} \int_0^\infty \tilde{g}(t) dt = \|h\|_{G(1)} \|g\|_{D(1)}.$$

Hence, $D(1) \cdot G(1) \subset L^1$ and

$$\|f\|_{L^1} \leq \inf\{\|g\|_{D(1)} \|h\|_{G(1)}: f = gh, g \in D(1), h \in G(1)\}.$$

This also means that $G(1) \subset D(1)'$ and $\|h\|_{D(1)'} \leq \|h\|_{G(1)}$. We show that we have in fact even equality. If $f \in D(1)'$, then

$$\begin{aligned} \frac{1}{x} \int_0^x |f(t)| dt &= \frac{1}{x} \int_0^1 \chi_{[0,x]}(t) |f(t)| dt \\ &\leq \frac{1}{x} \|\chi_{[0,x]}\|_{D(1)} \|f\|_{D(1)'} = \|f\|_{D(1)'}, \end{aligned}$$

for all $x > 0$, i.e., $f \in G(1)$ and so $D(1)' \subset G(1)$ with $\|f\|_{G(1)} \leq \|f\|_{D(1)'}$. Of course, $G(1)' = D(1)'' = D(1)$ since the norm of $D(1)$ has the Fatou property. Finally, if $f \in L^1$, then, by the Lozanovskii factorization theorem ([40, Theorem 6, p. 429]; cf. also [43, p. 185]), we can find $g \in D(1)$ and $h \in D(1)' = G(1)$ such that $f = g \cdot h$ and

$$\|g\|_{D(1)}\|h\|_{G(1)} = \|f\|_{L^1}.$$

This ends the proof of Proposition 2. \square

Remark 1. In particular, Proposition 2 shows that $(\text{Ces}_\infty[0, \infty))' = G(1)' = D(1)$. Thus, for the Cesàro function space on $[0, \infty)$ we get the result analogous to the Luxemburg–Zaanen theorem (cf. [42]): $(\text{Ces}_\infty[0, 1])' = \tilde{L}^1[0, 1]$, where $\|f\|_{\tilde{L}^1} = \|\tilde{f}\|_{L^1[0, 1]}$ with $\tilde{f}(x) = \text{ess sup}_{t \in [x, 1]} |f(t)|$.

Remark 2. For a positive weight function w and $1 \leq p < \infty$ let us define the weighted spaces $D(w, p)$ and $G(w, p)$ by the norms $\|f\|_{D(w, p)} = (\int_0^\infty \tilde{f}(x)^p \times w(x) dx)^{1/p}$, where $\tilde{f}(x) = \text{ess sup}_{t \in [x, \infty)} |f(t)|$, and $\|f\|_{G(w, p)} = \sup_{x>0} (\frac{1}{W(x)} \times \int_0^x |f(t)|^p dt)^{1/p}$, $W(x) = \int_0^x w(t) dt$, respectively. Proposition 2 is valid for weighted spaces: If $1 \leq p < \infty$, then $D(w, p) \cdot G(w, p) = L^p$ and $\|f\|_{L^p} = \inf\{\|g\|_{D(w, p)} \times \|h\|_{G(w, p)} : f = gh, g \in D(w, p), h \in G(w, p)\}$.

Proposition 3. Let $1 < p < \infty$. If $g \in (\text{Ces}_p)'$, then $\tilde{g}(x) = \text{ess sup}_{t \in [x, \infty)} |g(t)| \in (\text{Ces}_p)'$ and

$$\|\tilde{g}\|_{C(p)'} \leq 8\|g\|_{C(p)'}$$

Proof. Let $f \in \text{Ces}_p$, $f \geq 0$. Then $\int_0^x f(t) dt \rightarrow 0$ if $x \rightarrow 0^+$. Consider two cases:

(a) If $\int_0^\infty f(s) ds = \infty$, then we select a two-sided sequence $\{a_k\}_{k \in \mathbb{Z}}$ such that $0 \leq a_k < a_{k+1}$, $a_k \rightarrow \infty$ when $k \rightarrow \infty$ and

$$(6) \quad \int_{a_{k-1}}^{a_k} f(s) ds = 2^k, \quad k \in \mathbb{Z}.$$

(b) If $A = \int_0^\infty f(s) ds < \infty$, we find a one-sided sequence $\{a_k\}_{k \leq 0}$ such that $0 \leq a_k < a_{k+1}$, $a_0 = \infty$ and

$$(7) \quad \int_{a_{k-1}}^{a_k} f(s) ds = 2^{k-1} A, \quad k \leq 0.$$

By J let us denote either \mathbb{Z} or $\{k \in \mathbb{Z} : k \leq 0\}$ depending on which of the cases (a) or (b) we have, and let

$$\mathcal{P} = \left\{ k \in J: \text{ there is a set } A_k \subset [a_{k-1}, a_k) \text{ such that } m(A_k) > 0 \right. \\ \left. \text{and } |g(s)| \geq \frac{1}{2} \tilde{g}(a_{k-1}) \text{ for all } s \in A_k \right\}.$$

Note that $\mathcal{P} \neq \emptyset$. In fact, let $k \in J$ be arbitrary and let i be the first “time” such that $i \geq k$ and

$$m \left\{ s \in (a_{i-1}, a_i]: |g(s)| \geq \frac{1}{2} \tilde{g}(a_{k-1}) \right\} > 0.$$

Since $\tilde{g}(a_{i-1}) = \tilde{g}(a_{k-1})$, then $i \in \mathcal{P}$.

Let $\mathcal{P} = \{k_i\}_{i=l}^m$, where $k_i < k_j$ ($i < j$) and l may be $-\infty$. Moreover, it is easily seen that either $m = \infty$ and $k_i \rightarrow \infty$ when $i \rightarrow \infty$ (in the case (a)) or $k_m = 0$ and $t_{k_m} = \infty$ (in the case (b)).

Define the function

$$\bar{f}(t) = \sum_{i=l}^m \int_{\Delta_i} f(s) ds \frac{1}{m(A_{k_i})} \chi_{A_{k_i}}(t),$$

where $\Delta_i = (a_{k_i-1}, a_{k_i}]$, and estimate its norm in Ces_p .

Let $\bar{a} = \lim_{i \rightarrow -\infty} a_{k_i}$ if $l = -\infty$ and $\bar{a} = a_{k_l}$ if l is finite. If $\bar{a} > 0$, then $\bar{f}(t) = 0$ for all $t \in [0, \bar{a})$. Therefore

$$(8) \quad \int_0^x \bar{f}(t) dt = 0 \quad (0 < x \leq \bar{a}).$$

Suppose $x > \bar{a}$. Then either (1^o) $t \in \Delta_i$ for some i or (2^o) there is $i < m$ such that $t \in (a_{k_i}, a_{k_{i+1}-1}]$. In the first case, by (6) or (7) it yields that

$$\begin{aligned} \int_0^x \bar{f}(t) dt &= \sum_{j=l}^{i-1} \int_{\Delta_j} f(s) ds \frac{1}{m(A_{k_j})} m(A_{k_j}) \\ &\quad + \frac{m(A_{k_i} \cap (a_{k_i-1}, t])}{m(A_{k_i})} \int_{\Delta_i} f(s) ds \\ &\leq \int_0^{a_{k_i}} f(s) ds \leq 2 \int_0^x f(s) ds. \end{aligned}$$

Analogously, in the second case we have that

$$\int_0^x \bar{f}(t) dt \leq \int_0^{a_{k_i}} f(s) ds \leq \int_0^x f(s) ds.$$

The last inequalities and equality (8) show that

$$(9) \quad \|\bar{f}\|_{C(p)} \leq 2\|f\|_{C(p)}.$$

Moreover, for any i running from l to m we find that

$$(10) \quad \begin{aligned} \int_{\Delta_i} \bar{f}(t)|g(t)| dt &= \int_{A_{k_i}} \bar{f}(t)|g(t)| dt \geq \frac{1}{2} \tilde{g}(a_{k_i-1}) \int_{A_{k_i}} \bar{f}(t) dt \\ &= \frac{1}{2} \tilde{g}(a_{k_i-1}) \int_{\Delta_i} f(t) dt. \end{aligned}$$

Since \tilde{g} decreases, then (10) implies, in particular, that

$$(11) \quad \int_{\Delta_i} \bar{f}(t)|g(t)| dt \geq \frac{1}{2} \int_{\Delta_i} f(t) \tilde{g}(t) dt.$$

Note that, by definition of the set \mathcal{P} , it yields that $\tilde{g}(t) \leq \tilde{g}(a_{k_i-1})$ a.e. on the interval $(a_{k_{i-1}}, a_{k_i-1}]$ if $i > l$ and on the interval $(0, a_{k_l-1}]$ if l is finite. Moreover, taking into account (6) or (7) once again, we have that

$$\int_{a_{k_{i-1}}}^{a_{k_i-1}} f(s) ds \leq \int_{\Delta_i} f(s) ds \quad \text{if } i > l$$

and

$$\int_0^{a_{k_l-1}} f(s) ds \leq \int_{\Delta_l} f(s) ds \quad \text{if } l \text{ is finite.}$$

Therefore, by (10), it follows that

$$\begin{aligned} \int_{\Delta_i} \bar{f}(t)|g(t)| dt &\geq \frac{1}{2} \tilde{g}(a_{k_i-1}) \int_{\Delta_i} f(t) dt \geq \frac{1}{2} \tilde{g}(a_{k_i-1}) \int_{a_{k_{i-1}}}^{a_{k_i-1}} f(t) dt \\ &\geq \frac{1}{2} \int_{a_{k_{i-1}}}^{a_{k_i-1}} \tilde{g}(t) f(t) dt, \end{aligned}$$

where $a_{l-1} = 0$ if l is finite.

Since $f = 0$ a.e. on the interval $(0, \bar{a}]$, when $l = -\infty$ and $\bar{a} = \lim_{i \rightarrow -\infty} a_{k_i} > 0$, then, by summing the last inequalities and inequality (11) over all i , we get that

$$2 \int_0^\infty \bar{f}(t)|g(t)| dt \geq 2 \sum_{i=l}^m \int_{\Delta_i} \bar{f}(t)|g(t)| dt \geq \frac{1}{2} \int_0^\infty \tilde{g}(t) f(t) dt,$$

whence,

$$\int_0^\infty \tilde{g}(t)f(t) dt \leq 4 \int_0^\infty \tilde{f}(t)|g(t)| dt.$$

Combining the last inequality with (9), we obtain that

$$\begin{aligned} \|\tilde{g}\|_{C(p)'} &= \sup \left\{ \int_0^\infty \tilde{g}(t)f(t) dt : \|f\|_{C(p)} \leq 1 \right\} \\ &\leq 4 \sup \left\{ \int_0^\infty \tilde{f}(t)|g(t)| dt : \|f\|_{C(p)} \leq 1 \right\} \\ &\leq 4 \sup \left\{ \int_0^\infty \tilde{f}(t)|g(t)| dt : \|\tilde{f}\|_{C(p)} \leq 2 \right\} = 8\|g\|_{C(p)'} \end{aligned}$$

and the proof is complete. \square

Proof of Theorem 2. Firstly, we show that $D(p') \xhookrightarrow{1} (L^p \cdot G(p'))'$. In fact, let $f \in D(p')$ and $g \in L^p \cdot G(p')$, then $g = h \cdot k$ with $h \in L^p$ and $k \in G(p')$. By the Hölder–Rogers inequality and the imbedding $D(p') \cdot G(p') \xhookrightarrow{1} L^{p'}$ proved in Proposition 2 we obtain that

$$\|fg\|_{L^1} = \|fhk\|_{L^1} \leq \|h\|_{L^p} \|fk\|_{L^{p'}} \leq \|h\|_{L^p} \|k\|_{G(p')} \|f\|_{D(p')},$$

from which it follows that $D(p') \subset (L^p \cdot G(p'))'$ and $\|f\|_{(L^p \cdot G(p'))'} \leq \|f\|_{D(p')}$. Since, by Proposition 1 we have equality $\text{Ces}_p = L^p \cdot G(p')$, it follows that

$$D(p') \xhookrightarrow{p'} (\text{Ces}_p)'.$$

To prove the converse, take $f \in (\text{Ces}_p)'$. Since $\tilde{f} \geq |f|$ and $D(p')$ is a Banach lattice, then by Proposition 3, we may (and will) assume that f is a non-negative decreasing function on $(0, \infty)$, i.e., $f = \tilde{f}$. Then, by the Hardy inequality, we find that

$$\begin{aligned} \|f\|_{D(p')} &= \|f\|_{L^{p'}} = \sup \left\{ \int_0^\infty |f(x)g(x)| dx : \|g\|_{L^p} \leq 1 \right\} \\ &\leq p' \sup \left\{ \int_0^\infty |f(x)g(x)| dx : \|g\|_{C(p)} \leq 1 \right\} = p' \|f\|_{(\text{Ces}_p)'}. \end{aligned}$$

Therefore, $f \in D(p')$ and $(\text{Ces}_p)' \xhookrightarrow{8p'} D(p')$. \square

We describe the dual and Köthe dual of $\text{Ces}_p(I)$ for $1 < p < \infty$ in the case $I = [0, 1]$. Surprisingly this will have a different description than in the case $I = [0, \infty)$. For $p = \infty$ the space $\text{Ces}_\infty[0, 1]$ introduced by Korenblyum, Kreĭn and Levin [31] we denote by K and its separable part by K_0 .

As we already mentioned the Köthe dual space K' was found by Luxemburg and Zaanan [42]: $K' = \tilde{L}^1$ with equality of norms, where

$$\|f\|_{\tilde{L}^1} = \|\tilde{f}\|_{L^1}, \quad \text{with } \tilde{f}(x) = \text{ess sup}_{t \in [x, 1]} |f(t)|.$$

Earlier the dual space of K_0 was found by Tandori [56]: $(K_0)^* = \tilde{L}^1$ with equality of norms.

We will find the Köthe dual space $(\text{Ces}_p[0, 1])'$ for $1 < p < \infty$. Consider, for $1 < p < \infty$, a Banach ideal space $U(p)$ on $I = [0, 1]$ which norm is given by the formula

$$(12) \quad \|f\|_{U(p)} = \left\| \frac{1}{1-x^{1/(p-1)}} \tilde{f}(x) \right\|_{L^p} = \left[\int_0^1 \left(\frac{\tilde{f}(x)}{1-x^{1/(p-1)}} \right)^p dx \right]^{1/p},$$

where $\tilde{f}(x) = \text{ess sup}_{t \in [x, 1]} |f(t)|$.

Remark 3. Since $\min(1, p-1) \leq \frac{1-x}{1-x^{1/(p-1)}} \leq \max(1, p-1)$ for all $x \in (0, 1)$, then the norm (12) in $U(p)$ is equivalent to the norm

$$\|f\|_{U(p)}^0 = \left[\int_0^1 \left(\frac{\tilde{f}(x)}{1-x} \right)^p dx \right]^{1/p}.$$

Theorem 3. *If $1 < p < \infty$, then*

$$(13) \quad (\text{Ces}_p)^* = (\text{Ces}_p)' = U(p'), \quad p' = \frac{p}{p-1},$$

with equivalent norms.

Before the proof of this theorem we prove some auxiliary results of independent interest. First, for $1 < p < \infty$ we define the Banach ideal space $V(p)$ on $I = [0, 1]$ generated by the functional

$$(14) \quad \|f\|_{V(p)} = \sup_{0 < x \leq 1} \left[\frac{(1-x^{1/(p-1)})^{p-1}}{x} \int_0^x |f(t)|^p dt \right]^{1/p}.$$

Proposition 4. *If $1 < p < \infty$, then*

$$(15) \quad \text{Ces}_p \subset L^p \cdot V(p'), \quad p' = \frac{p}{p-1},$$

that is, if $f \in \text{Ces}_p$, then $f = gh$ with $g \in L^p$, $h \in V(p')$ and

$$(16) \quad \inf\{\|g\|_p \|h\|_{V(p')}: f = g \cdot h, g \in L^p, h \in V(p')\} \leq (p-1)^{1/p} \|f\|_{C(p)}.$$

Proof. The proof is analogous to the proof of Proposition 1 (for the case $I = [0, \infty)$) but we put details to see how the weight $w(x) = (1 - x^{p-1})^{1/(p-1)}$ appeared in the definition of the space $V(p')$. For $f \in \text{Ces}_p$, $f \neq 0$, define

$$k(x) = \int_x^1 u^{-p} \left(\int_0^u |f(t)| dt \right)^{p-1} du, \quad x \in [0, 1].$$

Then $k(x) > 0$, k is decreasing and, by the Hölder–Rogers inequality, we find that

$$\begin{aligned} k(x) &= \int_x^1 u^{-1} \left(\frac{1}{u} \int_0^u |f(t)| dt \right)^{p-1} du \\ &\leq \left(\int_x^1 u^{-p} du \right)^{1/p} \left(\int_x^1 \left(\frac{1}{u} \int_0^u |f(t)| dt \right)^p du \right)^{1/p'} \\ &\leq \frac{1}{(p-1)^{1/p}} \left(\frac{1-x^{p-1}}{x^{p-1}} \right)^{1/p} \|f\|_{C(p)}^{p-1}. \end{aligned}$$

Let

$$g(x) = (|f(x)|k(x))^{1/p} \operatorname{sgn} f(x) \quad \text{and} \quad h(x) = |f(x)|^{1/p'} k(x)^{-1/p}, \quad 0 < x < 1.$$

Then $f = g \cdot h$ and

$$\begin{aligned} \|g\|_p^p &= \int_0^1 |f(x)| \int_x^1 u^{-p} \left(\int_0^u |f(t)| dt \right)^{p-1} du dx \\ &= \int_0^1 u^{-p} \left(\int_0^u |f(t)| dt \right)^{p-1} \int_0^u |f(x)| dx du = \|f\|_{C(p)}^p, \end{aligned}$$

and, by the Hölder–Rogers inequality,

$$\begin{aligned} \left(\int_0^x |h(t)|^{p'} dt \right)^p &= \left(\int_0^x |f(t)|^{1/p'} |f(t)|^{1/p} k(t)^{-p'/p} dt \right)^p \\ &\leq \left(\int_0^x |f(t)| dt \right)^{p-1} \left(\int_0^x |f(t)| k(t)^{-p'} dt \right). \end{aligned}$$

Hence, by the above and using the fact that k is decreasing, we obtain that

$$\begin{aligned}
& \int_x^1 \left(s^{-1} \int_0^x |h(t)|^{p'} dt \right)^p ds \\
& \leq \int_x^1 s^{-p} \left[\left(\int_0^x |f(t)| dt \right)^{p-1} \int_0^x |f(t)| k(t)^{-p'} dt \right] ds \\
& \leq \int_x^1 s^{-p} \left[\left(\int_0^s |f(t)| dt \right)^{p-1} \int_0^x |f(t)| k(t)^{-p'} dt \right] ds \\
& = k(x) \int_0^x |f(t)| k(t)^{-p'} dt \\
& \leq \int_0^x |f(t)| k(t)^{1-p'} dt = \int_0^x |h(t)|^{p'} dt
\end{aligned}$$

or, equivalently,

$$\int_x^1 s^{-p} ds \left(\int_0^x |h(t)|^{p'} dt \right)^{p-1} \leq 1,$$

which means that

$$\left(\int_0^x |h(t)|^{p'} dt \right)^{p-1} \leq (p-1) \frac{x^{p-1}}{1-x^{p-1}}$$

and, thus,

$$\sup_{x>0} \frac{(1-x^{p-1})^{1/(p-1)}}{x} \int_0^x |h(t)|^{p'} dt \leq (p-1)^{1/(p-1)}$$

or $\|h\|_{V(p')} \leq (p-1)^{1/p}$. Summing up we have proved that $\text{Ces}_p \subset L^p \cdot V(p')$ and

$$\inf\{\|g\|_{L^p} \|h\|_{V(p')}: f = g \cdot h\} \leq (p-1)^{1/p} \|f\|_{C(p)}. \quad \square$$

Remark 4. In the above imbedding we cannot take instead of the space $V(p')$, where the weight $w(x) = (1-x^{p-1})^{1/(p-1)}$ appeared, the corresponding space without this weight, that is, the p' -convexification $K^{(p')}$ of K . This space is too small since if the imbedding $\text{Ces}_p[0, 1] \subset L^p \cdot K^{(p')}$ would be valid, then since $L^p \cdot K^{(p')} \subset L^p \cdot L^{p'} \subset L^1[0, 1]$ we will have a contradiction because $\text{Ces}_p[0, 1]$ is not embedded into $L^1[0, 1]$ (cf. Theorem 1(d)) and the problem is “near 1”, therefore this weight w is really needed in the imbedding (15).

Proposition 5. *If $1 < p < \infty$, then*

(a) $U(p) \cdot V(p) \subset L^p$ with

$$\|f\|_{L^p} \leq \inf\{\|g\|_{U(p)}\|h\|_{V(p)} : f = g \cdot h, g \in U(p), h \in V(p)\}.$$

(b) $U(p) \subset (V(p) \cdot L^{p'})'$ and $\|f\|_{(V(p) \cdot L^{p'})'} \leq \|f\|_{U(p)}$ for all $f \in U(p)$.

Proof. (a) Let $f = g \cdot h$, $g \in U(p)$, $h \in V(p)$. Since $|g| \leq \tilde{g}$ it follows that

$$(17) \quad \|f\|_{L^p}^p \leq \int_0^1 \tilde{g}(t)^p |h(t)|^p dt.$$

On the other hand, by the definition of the norm in $V(p)$ and using the equality

$$\frac{d}{dx} \left(\frac{x}{(1 - x^{1/(p-1)})^{p-1}} \right) = \frac{1}{(1 - x^{1/(p-1)})^p},$$

we obtain that

$$\begin{aligned} \int_0^x |h(t)|^p dt &\leq \|h\|_{V(p)}^p \frac{x}{(1 - x^{1/(p-1)})^{p-1}} \\ &= \|h\|_{V(p)}^p \int_0^x \frac{1}{(1 - t^{1/(p-1)})^p} dt \end{aligned}$$

for all $x \in (0, 1]$. Since \tilde{g}^p decreases, then, by [32, property 18⁰, p. 72], the last inequality implies that

$$\int_0^1 \tilde{g}(t)^p |h(t)|^p dt \leq \|h\|_{V(p)}^p \int_0^1 \left(\frac{\tilde{g}(t)}{1 - t^{1/(p-1)}} \right)^p dt.$$

Therefore, by (17), $f \in L^p$ and

$$\|f\|_{L^p} \leq \|g\|_{U(p)} \cdot \|h\|_{V(p)},$$

and the proof of (a) is complete.

(b) For any $f \in U(p)$ and $g \in V(p) \cdot L^{p'}$ we have $g = h \cdot k$ with $h \in V(p)$, $k \in L^{p'}$ and, by the Hölder–Rogers inequality and Proposition 5(a), we obtain that

$$\begin{aligned} \int_0^1 |fg| dx &= \int_0^1 |fhk| dx \leq \left(\int_0^1 |fh|^p dx \right)^{1/p} \left(\int_0^1 |k|^{p'} dx \right)^{1/p'} \\ &\leq \|f\|_{U(p)} \|h\|_{V(p)} \|k\|_{L^{p'}} = \|h\|_{V(p)} \|k\|_{L^{p'}} \|f\|_{U(p)} \end{aligned}$$

or $f \in (V(p) \cdot L^{p'})'$ and $\|f\|_{(V(p) \cdot L^{p'})'} \leq \|f\|_{U(p)}$. The proof of (b) is complete. \square

Proposition 6. Let $1 \leq p < \infty$. If $g \in (\text{Ces}_p)'$, then $\tilde{g}(x) = \text{ess sup}_{t \in [x, 1]} |g(t)| \in (\text{Ces}_p)'$ and

$$\|\tilde{g}\|_{C(p)'} \leq 8\|g\|_{C(p)'}$$

Proof. When $p = 1$, then the assertion is obvious since $\text{Ces}_1 = L^1(\ln 1/t)$ and $(\text{Ces}_1)' = L^\infty(\ln^{-1} 1/t)$. Let $p > 1$ and $f \in \text{Ces}_p$, $f \geq 0$. Consider two cases:

(a) If $\int_0^1 f(s) ds = \infty$, then we select a two-sided sequence $\{a_k\}_{k \in \mathbb{Z}}$ such that $0 \leq a_k < a_{k+1}$, $a_k \rightarrow 1$ when $k \rightarrow \infty$ and

$$(18) \quad \int_{a_{k-1}}^{a_k} f(s) ds = 2^k, \quad k \in \mathbb{Z}.$$

(b) If $A = \int_0^1 f(s) ds < \infty$, then we can find an one-sided sequence $\{a_k\}_{k \leq 0}$ such that $0 \leq a_k < a_{k+1}$, $a_0 = 1$ and

$$(19) \quad \int_{a_{k-1}}^{a_k} f(s) ds = 2^{k-1} A, \quad k \leq 0.$$

The remaining part of the proof is completely analogous to the proof of Proposition 3 so we omit the details. \square

Proof of Theorem 3. “*Imbedding \supset* ”. If $f \in U(p')$, then, by Proposition 5(b) and Proposition 4, we obtain that

$$U(p') \subset (V(p') \cdot L^p)' \subset (\text{Ces}_p)' \quad \text{and} \quad \|f\|_{C(p)'} \leq (p-1)^{1/p} \|f\|_{U(p')}.$$

“*Imbedding \subset* ”. Let $f \in (\text{Ces}_p)'$. Since $\tilde{f} \geq |f|$ and $U(p')$ is a Banach lattice, then by Proposition 6 we may (and we will) assume that f is a non-negative decreasing function on $(0, 1]$, i.e., $f = \tilde{f}$. Define the weight

$$w(x) = \chi_{[0, 1/2]}(x) + (1-x)\chi_{[1/2, 1]}(x), \quad 0 < x \leq 1.$$

Since $1-x \leq w(x) \leq 2(1-x)$ for $x \in (0, 1]$, then according to Remark 3 it is enough to prove that for some constant $A_p > 0$ we have that

$$(20) \quad \left\| \frac{f}{w} \right\|_{L^{p'}} = \left[\int_0^1 \left(\frac{f(x)}{w(x)} \right)^{p'} dx \right]^{1/p'} \leq A_p \|f\|_{C(p)'}$$

since

$$\begin{aligned} \|f\|_{U(p')} &= \left\| \frac{1}{1-x^{1/(p'-1)}} f(x) \right\|_{L^{p'}} \leq \max(1, p'-1) \left\| \frac{1}{1-x} f(x) \right\|_{L^{p'}} \\ &\leq 2 \max(1, p'-1) \|f/w\|_{L^{p'}}. \end{aligned}$$

We now prove that if $h \in L^p$, $h \geq 0$, then $h/w \in \text{Ces}_p$ and

$$(21) \quad \|h/w\|_{C(p)} \leq (p' + 2p)\|h\|_{L^p}.$$

To prove this we first show that the operator S_w defined by

$$S_w h(x) = \int_0^x \frac{h(t)}{w(t)} dt \quad (0 < x \leq 1)$$

is bounded in $L^p[0, 1]$ for $1 \leq p < \infty$. In fact, for $0 < x \leq 1/2$ we have that

$$S_w h(x) = \int_0^x h(t) dt = \int_{1-x}^1 h(1-t) dt \leq \int_{1-x}^1 \frac{h(1-t)}{t} dt$$

and for $1/2 \leq x \leq 1$

$$\begin{aligned} S_w h(x) &= \int_0^{1/2} h(t) dt + \int_{1/2}^x \frac{h(t)}{1-t} dt \\ &= \int_{1/2}^1 h(1-t) dt + \int_{1-x}^{1/2} \frac{h(1-t)}{t} dt \leq \int_{1-x}^1 \frac{h(1-t)}{t} dt. \end{aligned}$$

Thus,

$$S_w h(x) \leq H'(\bar{h})(1-x) \quad \text{for } 0 < x < 1,$$

where $\bar{h}(t) = h(1-t)$ and H' is the associated Hardy operator, i.e., $H'h(x) = \int_x^1 \frac{h(t)}{t} dt$. It is well known that H' is bounded in $L^p[0, 1]$ for $1 \leq p < \infty$ (cf. [32], pp. 138–139) and, thus,

$$\|S_w h\|_{L^p} \leq \|H'(\bar{h})\|_{L^p} \leq \|H'\| \|\bar{h}\|_{L^p} = \|H'\| \|h\|_{L^p}.$$

Since

$$\frac{1}{x} S_w h(x) = \frac{1}{x} \int_0^x \frac{h(t)}{w(t)} dt \leq \frac{1}{x} \int_0^x h(t) dt \chi_{[0, \frac{1}{2}]}(x) + 2S_w h(x) \chi_{[\frac{1}{2}, 1]}(x)$$

it follows that

$$\begin{aligned} \|h/w\|_{C(p)} &= \left\| \frac{1}{x} S_w h(x) \right\|_{L^p} \leq \|Hh\|_{L^p} + 2\|S_w h\|_{L^p} \\ &\leq p'\|h\|_{L^p} + 2p\|h\|_{L^p} = (p' + 2p)\|h\|_{L^p} \end{aligned}$$

and the estimate (21) is proved. Moreover, by using this fact we obtain that

$$\begin{aligned} \left\| \frac{f}{w} \right\|_{L^{p'}} &= \sup \left\{ \int_0^1 \frac{f(t)}{w(t)} h(t) dt : h \geq 0, \|h\|_{L^p} \leq 1 \right\} \\ &\leq \sup \left\{ \int_0^1 \frac{f(t)}{w(t)} h(t) dt : h \geq 0, \left\| \frac{h}{w} \right\|_{C(p)} \leq p' + 2p \right\} \\ &\leq (p' + 2p) \|f\|_{C(p')} \end{aligned}$$

and also the estimate (20) is proved, which shows that $(\text{Ces}_p)' \subset U(p')$ and for every $f \in (\text{Ces}_p)'$

$$\|f\|_{U(p')} \leq 16 \max(1, p' - 1)(p' + 2p) \|f\|_{C(p')},$$

and the proof is complete. \square

Remark 5. Let $1 < p < \infty$. The L^p spaces have the property that the restriction of $L^p[0, \infty)$ to $[0, 1]$ gives the space $L^p[0, 1]$. The situation is different for Cesàro function spaces. In fact, if $f \in \text{Ces}_p[0, \infty)$ and $\text{supp } f \subset [0, 1]$, then

$$\begin{aligned} \|f\|_{\text{Ces}_p[0, \infty)}^p &= \int_0^\infty \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \\ &= \int_0^1 \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx + \int_1^\infty \left(\frac{1}{x} \int_0^1 |f(t)| dt \right)^p dx \\ &= \|f\|_{\text{Ces}_p[0, 1]}^p + \frac{1}{p-1} \|f\|_{L^1[0, 1]}^p, \end{aligned}$$

which means that

$$\text{Ces}_p[0, \infty)|_{[0, 1]} = \text{Ces}_p[0, 1] \cap L^1[0, 1].$$

Therefore,

$$(\text{Ces}_p[0, 1] \cap L^1[0, 1])' = (\text{Ces}_p[0, \infty))'|_{[0, 1]} = D(p')|_{[0, 1]}$$

or

$$U(p') + L^\infty[0, 1] = D(p')|_{[0, 1]}.$$

The last equality can be easily verified. For example, for $f \in D(p')|_{[0, 1]}$ we can take as a decomposition $f = g + h$, $g \in U(p')$, $h \in L^\infty[0, 1]$ the functions

$$g(x) = (1 - x)f(x) \quad \text{and} \quad h(x) = xf(x), \quad x \in [0, 1].$$

Then $f = g + h$ and $\tilde{g}(x) = \text{ess sup}_{t \in [x, 1]} (1 - t)|f(t)| \leq (1 - x)\tilde{f}(x)$, which shows

that $g \in U(p')$ since $f \in D(p')$. Moreover,

$$\begin{aligned}\|h\|_\infty &= \operatorname{ess\,sup}_{x \in [0,1]} x|f(x)| \leq \operatorname{ess\,sup}_{x \in [0,1]} x\tilde{f}(x) \\ &\leq \|\tilde{f}\|_{L^1} \leq \|\tilde{f}\|_{L^{p'}} = \|f\|_{D(p')},\end{aligned}$$

so that $h \in L^\infty[0, 1]$.

5. ON p -CONCAVITY, TYPE AND COTYPE OF CESÀRO SEQUENCE AND FUNCTION SPACES

A Banach lattice X is said to be p -convex ($1 \leq p < \infty$) with constant $K \geq 1$, respectively q -concave ($1 \leq q < \infty$) with constant $L \geq 1$ if

$$\left\| \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} \right\| \leq K \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p},$$

respectively

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq L \left\| \left(\sum_{k=1}^n |x_k|^q \right)^{1/q} \right\|,$$

for every choice of vectors x_1, x_2, \dots, x_n in X .

Of course, every Banach lattice is 1-convex with constant 1. In particular, ces_p and $\operatorname{Ces}_p(I)$ are 1-convex with constant 1. The spaces $L^p(I)$ are p -convex and p -concave with constant 1.

If the above estimates hold for pairwise disjoint elements $\{x_k\}_{k=1}^n$ in X , that is,

$$\left\| \sum_{k=1}^n x_k \right\| \leq K \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p},$$

respectively

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq L \left\| \sum_{k=1}^n x_k \right\|,$$

then we say that X satisfies an *upper p -estimate* with constant K and a *lower q -estimate* with constant L , respectively. It is obvious that a p -convex (q -concave) Banach lattice satisfies upper p -estimate (lower q -estimate).

Let $r_n : [0, 1] \rightarrow \mathbb{R}, n \in \mathbb{N}$, be the Rademacher functions, that is, $r_n(t) = \operatorname{sign}(\sin 2^n \pi t)$. A Banach space X has *type* $1 \leq p \leq 2$ if there is a constant $K > 0$ such that, for any choice of finitely many vectors x_1, \dots, x_n from X ,

$$\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt \leq K \left(\sum_{k=1}^n \|x_k\|^p \right)^{1/p}.$$

A Banach space X has *cotype* $q \geq 2$ if there is a constant $K > 0$ such that, for any choice of finitely many vectors x_1, \dots, x_n from X ,

$$\left(\sum_{k=1}^n \|x_k\|^q \right)^{1/q} \leq K \int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\| dt.$$

In order to complete this definition for $q = \infty$ the left-hand side should be replaced by $\max_{1 \leq k \leq n} \|x_k\|$.

We say that the space X has *trivial type* or *trivial cotype*, if it does not have any type bigger than one or any finite cotype, respectively.

More information and connections among the above notions may be found in [17] and [38].

Theorem 4. *If $1 < p < \infty$, then $\text{Ces}_p(I)$ are p -concave with constant 1, that is,*

$$(22) \quad \left(\sum_{k=1}^n \|f_k\|_{C(p)}^p \right)^{1/p} \leq \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_{C(p)},$$

for all $f_1, f_2, \dots, f_n \in \text{Ces}_p(I)$.

Proof. Inequality (22) taken to the power p means that

$$\sum_{k=1}^n \int_I \left(\frac{1}{x} \int_0^x |f_k(t)| dt \right)^p dx \leq \int_I \left[\frac{1}{x} \int_0^x \left(\sum_{k=1}^n |f_k(t)|^p \right)^{1/p} dt \right]^p dx.$$

If we show that

$$\sum_{k=1}^n \left(\frac{1}{x} \int_0^x |f_k(t)| dt \right)^p \leq \left[\frac{1}{x} \int_0^x \left(\sum_{k=1}^n |f_k(t)|^p \right)^{1/p} dt \right]^p$$

for every $x \in I$, then we are done. The last estimate can also be written as

$$\left[\sum_{k=1}^n \left(\int_0^x |f_k(t)| dt \right)^p \right]^{1/p} \leq \int_0^x \left(\sum_{k=1}^n |f_k(t)|^p \right)^{1/p} dt,$$

which is the p -concavity of $L^1[0, x]$ for every $x \in I$.

It is clear that $L^1(J)$, $J = J_x = [0, x]$ is 1-convex with constant 1 and it is well known that then $L^1(J)$ is p -concave with constant 1 (cf. [38, Proposition 1.d.5] or [44, Theorem 4.3]). We can also prove this fact directly as in [44, Theorem 4.3]: by the Hölder–Rogers inequality for $t \in J$ it yields that

$$\sum_{k=1}^n |f_k(t)| |a_k| \leq \left(\sum_{k=1}^n |f_k(t)|^p \right)^{1/p} \|\{a_k\}\|_{p'}$$

and, by integrating over J ,

$$\begin{aligned} \int_J \sum_{k=1}^n |f_k(t)| |a_k| dt &\leq \| \{a_k\} \|_{p'} \int_I \left(\sum_{k=1}^n |f_k(t)|^p \right)^{1/p} dt \\ &= \| \{a_k\} \|_{p'} \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_{L^1(J)}. \end{aligned}$$

Taking the supremum over all $\{a_k\}$ such that $\| \{a_k\} \|_{p'} \leq 1$ we obtain, by the Landau theorem,

$$\begin{aligned} &\sup \left\{ \int_J \sum_{k=1}^n |f_k(t)| |a_k| dt : \| \{a_k\} \|_{p'} \leq 1 \right\} \\ &= \sup \left\{ \sum_{k=1}^n |a_k| \int_J |f_k(t)| dt : \| \{a_k\} \|_{p'} \leq 1 \right\} \\ &= \left\| \left\{ \int_J |f_k(t)| dt \right\} \right\|_p = \left[\sum_{k=1}^n \left(\int_J |f_k(t)| dt \right)^p \right]^{1/p} \\ &= \left(\sum_{k=1}^n \|f_k\|_{L^1(J)}^p \right)^{1/p}. \end{aligned}$$

Thus,

$$\left(\sum_{k=1}^n \|f_k\|_{L^1(J)}^p \right)^{1/p} \leq \left\| \left(\sum_{k=1}^n |f_k|^p \right)^{1/p} \right\|_{L^1(J)},$$

and putting these facts together we obtain the estimate (22). \square

Theorem 5. *If $1 < p < \infty$, then the space $\text{Ces}_p(I)$ has trivial type and cotype $\max(p, 2)$. The space $\text{Ces}_\infty(I)$ has trivial type and trivial cotype.*

Proof. Let $1 < p < \infty$. The space $\text{Ces}_p(I)$ contains a copy of $L^1(I)$ (cf. [4], Lemma 1 for $I = [0, 1]$ and Theorem 2 for $I = [0, \infty)$) which implies that $\text{Ces}_p(I)$ has trivial type.

On the other hand, since, by Theorem 4 the space $\text{Ces}_p(I)$ is p -concave, then by a well-known theorem (cf. Lindenstrauss and Tzafriri [38, p. 100]) it has cotype $\max(p, 2)$. The fact that this space has no smaller cotype follows, for example, from Theorem 6 showing that $\text{Ces}_p(I)$ contains an isomorphic copy of l^p and the fact that the space l^p has cotype $\max(p, 2)$ and this value is the best possible (cf. [38, p. 73] or [44, pp. 91–94]).

For $p = \infty$ the space $\text{Ces}_\infty(I)$ has no absolutely continuous norm and, by the Lozanovskii theorem (see [40, Theorem 5, p. 65]; cf. also [28, Theorem 4 in X.4])

and [59, Theorem 4.1]), it contains an isomorphic copy of l^∞ , therefore it has trivial type and trivial cotype. The proof is complete. \square

Remark 6. Similarly as in Theorem 4 we can prove that the Cesàro sequence spaces ces_p are p -concave with constant 1 since l^1 is p -concave with constant 1. Moreover, similarly as in Theorem 5 we can obtain that the Cesàro sequence spaces ces_p have trivial type and cotype $\max(p, 2)$ for $1 < p < \infty$. Also ces_∞ has trivial type and trivial cotype.

6. COPIES OF l^p SPACES IN THE CESÀRO FUNCTION SPACES Ces_p

The Cesàro function space $\text{Ces}_p(I)$ contains a copy of $L^1(I)$ and as we will see in the next theorem also complemented copies of l^p .

Theorem 6. *If $1 < p < \infty$, then $\text{Ces}_p(I)$ contains an order isomorphic and complemented copy of l^p .*

Proof. Let $I = [0, 1]$. We shall construct a sequence $\{f_n\}_{n=1}^\infty \subset \text{Ces}_p[0, 1]$ with disjoint supports which spans an isomorphic copy of l^p in $\text{Ces}_p[0, 1]$ and the closed linear span $[f_n]_{\text{Ces}_p}$ is complemented in $\text{Ces}_p[0, 1]$. Let us denote

$$f_n = \chi_{[2^{-n-1}, 2^{-n}]} \quad \text{and} \quad F_n(t) = \frac{1}{t} \int_0^t f_n(s) ds, \quad n = 1, 2, \dots$$

Since

$$F_n(t) = \begin{cases} 0, & \text{if } 0 < t \leq 2^{-n-1}, \\ 1 - \frac{1}{2^{n+1}t}, & \text{if } 2^{-n-1} \leq t \leq 2^{-n}, \\ \frac{1}{2^{n+1}t}, & \text{if } t \geq 2^{-n}, \end{cases}$$

it follows that

$$\|f_n\|_{C(p)}^p = \|F_n\|_{L^p}^p = \int_{2^{-n-1}}^{2^{-n}} \left(1 - \frac{1}{2^{n+1}t}\right)^p dt + 2^{-(n+1)p} \frac{2^{n(p-1)} - 1}{p-1}.$$

Note that the first term in the above sum is not bigger than 2^{-p-n-1} and the second one satisfies the inequalities

$$\frac{1 - 2^{-p+1}}{p-1} 2^{-p-n} \leq 2^{-(n+1)p} \frac{2^{n(p-1)} - 1}{p-1} \leq \frac{2^{-p-n}}{p-1}.$$

Therefore,

$$(23) \quad \|f_n\|_{C(p)} \approx \|f_n\|_{L^p} \approx 2^{-n/p}$$

with constants which depend only on p . If

$$\tilde{f}_n = \frac{f_n}{\|f_n\|_{C(p)}}, \quad n = 1, 2, \dots,$$

then

$$1 = \|\tilde{f}_n\|_{C(p)} \approx \|\tilde{f}_n\|_{L^p}, \quad n \in \mathbb{N}.$$

Let us denote

$$x(t) = \sum_{n=1}^{\infty} \alpha_n \tilde{f}_n, \quad \alpha_n \in \mathbb{R}.$$

Since \tilde{f}_n are disjoint functions we may assume that $\alpha_n \geq 0$. By Theorem 1(c) (the Hardy inequality) and the above equivalence

$$\|x\|_{C(p)} \leq \frac{p}{p-1} \|x\|_{L^p} = \frac{p}{p-1} \left(\sum_{n=1}^{\infty} \alpha_n^p \|\tilde{f}_n\|_{L^p}^p \right)^{1/p} \leq C_p \left(\sum_{n=1}^{\infty} \alpha_n^p \right)^{1/p}.$$

On the other hand, by Theorem 4, for any $n \in \mathbb{N}$,

$$\left\| \sum_{k=1}^n \alpha_k \tilde{f}_k \right\|_{C(p)} \geq \left(\sum_{k=1}^n \|\alpha_k \tilde{f}_k\|_{C(p)}^p \right)^{1/p} = \left(\sum_{k=1}^n \alpha_k^p \right)^{1/p}$$

and passing to the limit as $n \rightarrow \infty$ we arrive at the inequality

$$\|x\|_{C(p)} \geq \left(\sum_{k=1}^{\infty} \alpha_k^p \right)^{1/p} \approx \|x\|_{L^p},$$

which together with estimation from above gives us that

$$(24) \quad [\tilde{f}_n]_{\text{Ces}_p} \simeq [\tilde{f}_n]_{L^p} \simeq l^p.$$

Next, we prove that $[\tilde{f}_n]_{\text{Ces}_p}$ is complemented in Ces_p for $1 < p < \infty$. Let $x \in \text{Ces}_p$, $x \geq 0$ and $\text{supp } x \subset [2^{-n-1}, 2^{-n}]$, $n \in \mathbb{N}$. Then

$$\frac{1}{t} \int_0^t x(s) ds = \frac{1}{t} \int_{2^{-n-1}}^t x(s) ds \chi_{[2^{-n-1}, 2^{-n}]}(t) + \frac{1}{t} \|x\|_{L^1} \chi_{[2^{-n}, 1]}(t)$$

and

$$\|x\|_{C(p)}^p = \int_{2^{-n-1}}^{2^{-n}} \left(\frac{1}{t} \int_{2^{-n-1}}^t x(s) ds \right)^p dt + \|x\|_{L^1}^p \int_{2^{-n}}^1 t^{-p} dt.$$

The first term in the last sum is not bigger than

$$\|x\|_{L^1}^p \int_{2^{-n-1}}^{2^{-n}} t^{-p} dt = \frac{2^{p-1} - 1}{p - 1} 2^{n(p-1)} \|x\|_{L^1},$$

and the second one is equal to

$$\|x\|_{L^1}^p \frac{2^{n(p-1)} - 1}{p - 1}.$$

Therefore,

$$(25) \quad \|x\|_{C(p)} \approx \|x\|_{L^1} 2^{n(1-1/p)}, \quad n = 1, 2, \dots,$$

with constants which depend only on p . We consider the orthogonal projector

$$(26) \quad Tx(t) := \sum_{k=1}^{\infty} 2^{k+1} \int_{2^{-k-1}}^{2^{-k}} x(s) ds \chi_{[2^{-k-1}, 2^{-k}]}(t)$$

and prove that it is bounded in Ces_p .

For arbitrary $x \in \text{Ces}_p$, $x \geq 0$ we set $x_k = x \chi_{[2^{-k-1}, 2^{-k}]} (k = 1, 2, \dots)$. Since

$$Tx_k = \|x_k\|_{L^1} 2^{k+1} \chi_{[2^{-k-1}, 2^{-k}]},$$

then (23) and (25) imply that

$$\|Tx_k\|_{C(p)} = \|x_k\|_{L^1} 2^{k+1} \|f_k\|_{C(p)} \leq B \|x_k\|_{L^1} 2^{k+1} 2^{-k/p} \leq C \|x_k\|_{C(p)}.$$

Therefore, by (24) and Theorem 4, we have that

$$\begin{aligned} \|Tx\|_{C(p)} &\leq C' \left(\sum_{k=1}^{\infty} \|Tx_k\|_{C(p)}^p \right)^{1/p} \leq C' C \left(\sum_{k=1}^{\infty} \|x_k\|_{C(p)}^p \right)^{1/p} \\ &\leq C' C \left\| \sum_{k=1}^{\infty} x_k \right\|_{C(p)} = C' C \|x\|_{C(p)}, \end{aligned}$$

and the proof of the boundedness of T in Ces_p is complete. Since the image of T coincides with $[x_n]_{\text{Ces}_p}$, then Theorem 6 is proved. \square

The above theorem shows that the Cesàro function spaces $\text{Ces}_p[0, 1]$ behave “near zero” similar to the l^p spaces. The authors proved in [4] that “in the middle” Cesàro function spaces $\text{Ces}_p(I)$ contain an asymptotically isometric copy of l^1 , that is, there exist a sequence $\{\varepsilon_n\} \subset (0, 1)$, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and a sequence of functions $\{f_n\} \subset \text{Ces}_p[0, 1]$ such that, for arbitrary $\{\alpha_n\} \in l^1$, we have that

$$(27) \quad \sum_{n=1}^{\infty} (1 - \varepsilon_n) |\alpha_n| \leq \left\| \sum_{n=1}^{\infty} \alpha_n f_n \right\|_{C(p)} \leq \sum_{n=1}^{\infty} |\alpha_n|.$$

Consequently, these spaces are not reflexive and do not have the fixed point property. This is a big difference with the Cesàro sequence spaces ces_p , which for $1 < p < \infty$ are reflexive and have the fixed point property.

Let us recall that a Banach space X has the *Dunford–Pettis property* if $x_n \rightarrow 0$ weakly in X and $f_n \rightarrow 0$ weakly in the dual space X^* imply $f_n(x_n) \rightarrow 0$. The classical examples of Banach spaces with the Dunford–Pettis property are the AL-spaces and AM-spaces. It is clear that if X^* has the Dunford–Pettis property, then X has itself this property (cf. [2, pp. 334–336]). Of course, the Cesàro sequence spaces ces_p , $1 < p < \infty$, as reflexive spaces do not have the Dunford–Pettis property.

Corollary 1. *If $1 < p < \infty$, then $\text{Ces}_p(I)$ do not have the Dunford–Pettis property.*

Proof. By Theorem 6, $\text{Ces}_p(I)$ contains a complemented copy of l^p and l^p do not have the Dunford–Pettis property. On the other hand, it is easy to show that, if a Banach space has the Dunford–Pettis property, then its complemented subspace has also the Dunford–Pettis property (cf. Wnuk [58, Lemma 1(i)] or [23, Proposition 11.37]). Thus, $\text{Ces}_p(I)$ do not have the Dunford–Pettis property. \square

As it was mentioned before the Cesàro sequence spaces ces_p are not isomorphic to the l^q space for any $1 \leq q \leq \infty$. An analogous theorem is true for Cesàro function spaces.

Theorem 7. *If $1 < p \leq \infty$, then $\text{Ces}_p(I)$ are not isomorphic to any $L^q(I)$ space for any $1 \leq q \leq \infty$.*

Proof. If $1 < q < \infty$, then $\text{Ces}_p(I)$ has trivial type but $L^q(I)$ has type $\min(q, 2) > 1$ and therefore they cannot be isomorphic. The space $\text{Ces}_p(I)$ for $1 < p < \infty$ is not isomorphic to $L^1(I)$ since $L^1(I)$ has the Dunford–Pettis property but $\text{Ces}_p(I)$, as we have seen in Corollary 1, do not have the Dunford–Pettis property. Also $\text{Ces}_p(I)$ for $1 < p < \infty$ is not isomorphic to $L^\infty(I)$ since the first space is separable and the second one is non-separable.

It only remains to show that $\text{Ces}_\infty(I)$ is not isomorphic to $L^\infty(I)$. Since, by Pełczyński theorem $L^\infty(I)$ is isomorphic to ℓ^∞ (cf. Albiac and Kalton [1, Theorem 4.3.10]), therefore it is enough to show that $\text{Ces}_\infty(I)$ is not isomorphic to ℓ^∞ .

We show this for $K = \text{Ces}_\infty[0, 1]$ since for the case of $\text{Ces}_\infty(0, \infty)$ the proof is similar. For fixed $a \in (0, 1)$ define a continuous projection $P : K \rightarrow K$ by $Pf = f \chi_{[a, 1]}$. Then

$$\begin{aligned} \int_a^1 |Pf(t)| dt &\leq \int_0^1 |Pf(t)| dt \leq \|Pf\|_K = \sup_{0 < x \leq 1} \frac{1}{x} \int_0^x |f(t) \chi_{[a, 1]}(t)| dt \\ &= \sup_{a \leq x \leq 1} \frac{1}{x} \int_a^x |f(t) \chi_{[a, 1]}(t)| dt \leq \frac{1}{a} \int_a^1 |Pf(t)| dt. \end{aligned}$$

Hence, $P(K)$ is isomorphic to $L^1[a, 1]$, i.e., K contains a complemented copy of a separable space while no separable subspace of ℓ^∞ is complemented in ℓ^∞ because ℓ^∞ is prime, that is, every infinite dimensional complemented subspace of ℓ^∞ is isomorphic to ℓ^∞ (see Lindenstrauss and Tzafriri [37, Theorem 2.a.7], or Albiac and Kalton [1, Theorem 5.6.5]). Therefore, K and ℓ^∞ are not isomorphic. \square

7. ON THE WEAK BANACH-SAKS PROPERTY OF THE CESÀRO FUNCTION SPACES

Let us recall that a Banach space X is said to have the *weak Banach-Saks property* if every weakly null sequence in X , say (x_n) , contains a subsequence (x_{n_k}) whose first arithmetical means converge strongly to zero, that is,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \left\| \sum_{k=1}^m x_{n_k} \right\| = 0.$$

It is known that uniformly convex spaces, c_0 , l^1 and L^1 have the weak Banach-Saks property, whereas $C[0, 1]$ and l^∞ do not have. We should mention that the result on L^1 space, proved by Szlenk [55] in 1965, was a very important breakthrough in studying of the weak Banach-Saks property.

In 1982, Rakov [48, Theorem 1] proved that a Banach space with non-trivial type (or equivalently B -convex) has the weak Banach-Saks property (cf. also [57, Theorem 1]). Recently Dodds, Semenov and Sukochev [19] investigated the weak Banach-Saks property of rearrangement invariant spaces and Astashkin and Sukochev [6] have got a complete description of Marcinkiewicz spaces with the latter property.

The spaces $\text{Ces}_p[0, 1]$ for $1 \leq p < \infty$ are neither B -convex (they have trivial type) nor rearrangement invariant. Nevertheless, we will prove that Ces_p for all $1 \leq p < \infty$ have the weak Banach-Saks property.

Theorem 8. *If $1 \leq p < \infty$, then the Cesàro function space $\text{Ces}_p[0, 1]$ has the weak Banach-Saks property.*

We begin with some auxiliary notation and results.

If $I = [a, b]$ and $J = [c, d]$ are two closed intervals, then we write $I < J$ if $b \leq c$. Let $\{I_n\}_{n=1}^\infty$ be a sequence of closed intervals $I_n = [a_n, b_n] \subset [0, 1]$. Then $I_n \rightarrow 0$ means that $I_1 > I_2 > \dots$ and $b_n \rightarrow 0^+$. Analogously, $I_n \rightarrow 1$ means that $I_1 < I_2 < \dots$ and $a_n \rightarrow 1^-$. Moreover, in what follows $\text{supp } f = \{t: f(t) \neq 0\}$.

Lemma 1 (Weakly null sequences in $\text{Ces}_p[0, 1]$, $1 < p < \infty$). *Let $\{x_n\}_{n=1}^\infty \subset \text{Ces}_p$. Then $x_n \xrightarrow{w} 0$ in Ces_p if and only if*

(a) *there exists a constant $M > 0$ such that $\|x_n\|_{C(p)} \leq M$ for all $n = 1, 2, \dots$,*

and

(b) *for every set $A \subset [0, 1]$ such that $A \subset [h, 1 - h]$ for some $h \in (0, \frac{1}{2})$ we have $\int_A x_n(t) dt \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. It is enough to check that the set of all functions of the form

$$(28) \quad a(t) = \sum_{k=1}^n a_k \chi_{A_k}(t),$$

where $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $A_k \subset [0, 1]$ are pairwise disjoint sets such that $A_k \subset [h, 1 - h]$ for some $h \in (0, \frac{1}{2})$, is dense in the space $U(p') = (\text{Ces}_p)^* = (\text{Ces}_p)'$, $p' = \frac{p}{p-1}$, with the norm

$$\|y\|_{U(p')} = \left(\int_0^1 \left(\frac{\tilde{y}(t)}{1-t} \right)^{p'} dt \right)^{1/p'}, \quad \tilde{y}(t) = \text{ess sup}_{s \in [t, 1]} |y(s)|.$$

Let $y \in U(p')$ and $\varepsilon > 0$. Note that $\tilde{y}(1^-) = \lim_{t \rightarrow 1^-} \tilde{y}(t) = 0$. In fact, if $\tilde{y}(t) \geq c > 0$ ($0 < t < 1$), then since $p' > 1$ we have that $\|y\|_{U(p')}^{p'} \geq c \int_0^1 \frac{1}{(1-t)^{p'}} dt = \infty$. Therefore, we may choose $\delta \in (0, 1)$ and $h \in (0, \delta)$ so that

$$(29) \quad \max \left(\int_0^\delta \left(\frac{\tilde{y}(t)}{1-t} \right)^{p'} dt, \int_{1-\delta}^1 \left(\frac{\tilde{y}(t)}{1-t} \right)^{p'} dt \right) \leq \varepsilon^{p'}$$

and

$$(30) \quad \tilde{y}(1-h) \leq \varepsilon \cdot \left(\frac{p'-1}{\delta^{1-p'}-1} \right)^{1/p'}.$$

Since $y \in U(p')$, then $\tilde{y}(t)$ is finite for every $t \in (0, 1)$ which implies that $y(t)$ is a bounded measurable function on the interval $[h, 1-h]$. Therefore, there exists a function $a(t)$ of the form (28) such that $\text{supp } a \subset [h, 1-h]$ and

$$(31) \quad \|(y-a)\chi_{[h, 1-h]}\|_{L^\infty} \leq \varepsilon \cdot \left(\frac{p'-1}{h^{1-p'}-1} \right)^{1/p'}.$$

By the triangle inequality we have that

$$(32) \quad \begin{aligned} & \|y-a\|_{U(p')} \\ & \leq \|y\chi_{[0, h]}\|_{U(p')} + \|(y-a)\chi_{[h, 1-h]}\|_{U(p')} + \|y\chi_{[1-h, 1]}\|_{U(p')}, \end{aligned}$$

and let us estimate each of the three terms separately. At first, since $0 < h < \delta$, then, by (29),

$$(33) \quad \|y\chi_{[0, h]}\|_{U(p')}^{p'} \leq \int_0^\delta \left(\frac{\tilde{y}(t)}{1-t} \right)^{p'} dt \leq \varepsilon^{p'}.$$

Next, (31) implies

$$(34) \quad \|(y-a)\chi_{[h, 1-h]}\|_{U(p')}^{p'} \leq \int_0^{1-h} \frac{dt}{(1-t)^{p'}} \cdot \varepsilon^{p'} \cdot \left(\frac{p'-1}{h^{1-p'}-1} \right) = \varepsilon^{p'}.$$

Finally, (30) and (29) imply that

$$(35) \quad \|y\chi_{[1-h,1]}\|_{U(p')}^{p'} \leq \int_0^{1-\delta} \left(\frac{\tilde{y}(1-h)}{1-t} \right)^{p'} dt + \int_{1-\delta}^1 \left(\frac{\tilde{y}(t)}{1-t} \right)^{p'} dt \leq 2\varepsilon^{p'}.$$

Thus, by (32)–(35), we have that $\|y - a\|_{U(p')} \leq 4^{1/p'} \varepsilon$, and the proof is complete. \square

Corollary 2. *Let $\{I_n\}_{n=1}^\infty$ be a sequence of intervals from $[0, 1]$ such that either $I_n \rightarrow 0$ or $I_n \rightarrow 1$. Then, for every $p \in (1, \infty)$, we have $\frac{\chi_{I_n}}{\|\chi_{I_n}\|_{C(p)}} \xrightarrow{w} 0$ in $\text{Ces}_p[0, 1]$.*

Following Kadec and Pełczyński [27] (see also [46] and [47]) we will use the following notation: Let X be a Banach function lattice on $[0, 1]$. For every $x \in X$ and $\alpha \in (0, 1]$ we set

$$\eta(x, \alpha) = \sup_{A \subset [0, 1], m(A) = \alpha} \|x\chi_A\|_X.$$

Moreover, if $K \subset X$, then

$$\eta(K, \alpha) = \sup_{x \in K} \eta(x, \alpha), \quad \eta(K, 0^+) = \lim_{\alpha \rightarrow 0^+} \eta(K, \alpha).$$

Lemma 2. *If a Banach function lattice X on $[0, 1]$ satisfies a lower p -estimate ($1 \leq p < \infty$) with constant one, then for any disjointly supported $x, y \in X$ and $\alpha > 0, \beta > 0$ we have that*

$$\eta(x + y, \alpha + \beta)^p \geq \eta(x, \alpha)^p + \eta(y, \beta)^p.$$

Proof. For any $\varepsilon > 0$ choose the sets A and B from $[0, 1]$ such that $A \subset \text{supp } x$, $B \subset \text{supp } y$, $m(A) \leq \alpha$, $m(B) \leq \beta$, and

$$\|x\chi_A\|_X^p \geq \eta(x, \alpha)^p - \varepsilon, \quad \|y\chi_B\|_X^p \geq \eta(y, \beta)^p - \varepsilon.$$

Since $m(A \cup B) \leq \alpha + \beta$ and X satisfies a lower p -estimate with constant one it follows that

$$\begin{aligned} \eta(x + y, \alpha + \beta) &\geq \|(x + y)\chi_{A \cup B}\|_X = \|x\chi_A + y\chi_B\|_X \\ &\geq (\|x\chi_A\|_X^p + \|y\chi_B\|_X^p)^{1/p} \\ &\geq (\eta(x, \alpha)^p + \eta(y, \beta)^p - 2\varepsilon)^{1/p}, \end{aligned}$$

and the proof of the lemma follows by letting $\varepsilon \rightarrow 0^+$. \square

Let X be a Banach function lattice on $[0, 1]$ and a set $K \subset X$. We say that K consists of elements having *equicontinuous norms* in X if

$$\lim_{A \subset [0, 1], m(A) \rightarrow 0} \sup_{x \in K} \|x\chi_A\|_X = 0.$$

An important tool in the proof of Theorem 8 will be the following assertion:

Proposition 7 (Subsequence splitting property). *Let $1 < p < \infty$, $\{x_n\}_{n=1}^\infty \subset \text{Ces}_p[0, 1]$, $\|x_n\|_{C(p)} = 1$ and $x_n \xrightarrow{w} 0$ in $\text{Ces}_p[0, 1]$. Then there exists a subsequence $\{x'_n\} \subset \{x_n\}$ such that*

$$x'_n = y_n + z_n, \quad n = 1, 2, \dots,$$

where $\{y_n\}_{n=1}^\infty$ consists of elements having equicontinuous norms in Ces_p and $\text{supp } z_n \subset I'_n \cup I''_n$ with $\{I'_n, I''_n\}_{n=1}^\infty$ being a sequence of pairwise disjoint intervals from $[0, 1]$ such that $I'_n \rightarrow 0$, $I''_n \rightarrow 1$. Moreover, $y_n \xrightarrow{w} 0$, $z_n \xrightarrow{w} 0$ in Ces_p .

Proof. We set $\eta_0 = \eta(\{x_n\}, 0^+)$. If $\eta_0 = 0$, then the sequence $\{x_n\}$ consists of elements with equicontinuous norms in Ces_p and we have nothing to prove. Therefore, assume that $\eta_0 > 0$. By the definition of η_0 , there exists a sequence of sets $A_n \subset [0, 1]$, $m(A_n) = \alpha_n \rightarrow 0$ and a subsequence of $\{x_n\}$ (which will be denoted also by $\{x_n\}$) such that for all $n \in \mathbb{N}$

$$(36) \quad \|x_n \chi_{A_n}\|_{C(p)} \geq \eta_0 - \frac{1}{n}.$$

Let us denote

$$(37) \quad u_n = x_n \chi_{A_n} \quad \text{and} \quad v_n = x_n - u_n.$$

Since $\text{Ces}_p[0, 1]$ is p -concave with constant one, then, by Lemma 2, it yields that

$$\eta(v_n, \alpha)^p \leq \eta(x_n, \alpha + \alpha_n)^p - \eta(u_n, \alpha_n)^p \leq \eta(x_n, \alpha + \alpha_n)^p - \left(\eta_0 - \frac{1}{n}\right)^p.$$

Hence, for $0 < \alpha \leq 1/2$ we have that

$$\limsup_{n \rightarrow \infty} \eta(v_n, \alpha)^p \leq \eta(\{x_n\}, 2\alpha)^p - \eta_0^p.$$

Since Ces_p is a separable space the last inequality implies that

$$(38) \quad \eta(\{v_n\}, 0^+) = 0,$$

that is, $\{v_n\}$ consists of elements with equicontinuous norms in Ces_p .

According to Lemma 1, for every $h \in (0, \frac{1}{2})$,

$$(39) \quad x_n \chi_{[h, 1-h]} \xrightarrow{w} 0 \quad \text{in } \text{Ces}_p.$$

Therefore, since $\text{Ces}_p[0, 1]_{[h, 1-h]} = L^1[h, 1-h]$ with equivalent norms (see [4, Lemma 1]) we have that $x_n \chi_{[h, 1-h]} \xrightarrow{w} 0$ in L^1 . Moreover, since $\eta(\{v_n\}, 0^+) = 0$ it follows that

$$\eta_{L^1}(\{v_n \chi_{[h, 1-h]}\}, 0^+) = 0$$

(where η_{L^1} is calculated in the space L^1) and

$$\|v_n \chi_{[h, 1-h]}\|_{L^1} \leq C \|v_n \chi_{[h, 1-h]}\|_{C(p)} \leq C.$$

Thus, by the classical Dunford–Pettis criterion (see, for example, [22, Theorem 4.21.2] or [1, Theorem 5.2.9]), the sequence $\{v_n \chi_{[h, 1-h]}\}_{n=1}^\infty$ is a relatively weakly compact subset of L^1 and, hence, simultaneously in Ces_p . Therefore, there is a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $v_{n_k} \chi_{[h, 1-h]} \xrightarrow{w} v$, where $v \in \text{Ces}_p$. By combining the last mentioned facts with (39) and with the equality $x_n = u_n + v_n$, we get that $u_{n_k} \chi_{[h, 1-h]} \xrightarrow{w} -v$ in Ces_p , and, hence, in L^1 . Taking into account the definition of u_n (see (36) and (37)) and using again the Dunford–Pettis criterion we conclude that for every $h \in (0, 1/2)$ there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ (depending on h) such that

$$\|u_{n_k} \chi_{[h, 1-h]}\|_{C(p)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Since Ces_p is a separable space, then by a standard procedure, we may choose a subsequence of $\{u_{n_k}\}$ (denote it again by $\{u_{n_k}\}$) and pairwise disjoint intervals $\{I'_k, I''_k\}_{k \in \mathbb{N}}$, $I'_k \rightarrow 0$, $I''_k \rightarrow 1$ such that

$$(40) \quad \|u_{n_k} \chi_{[0, 1] \setminus (I'_k \cup I''_k)}\|_{C(p)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Setting $x'_k = y_k + z_k$, with

$$y_k = v_{n_k} + u_{n_k} \chi_{[0, 1] \setminus (I'_k \cup I''_k)}, \quad z_k = u_{n_k} \chi_{I'_k \cup I''_k},$$

we see that, by (38) and (40), this representation satisfies all conditions. In particular, according to Lemma 1, we have that $z_k \xrightarrow{w} 0$ and $y_k \xrightarrow{w} 0$. The proof is complete. \square

Now, we may proceed with the proof of Theorem 8.

Proof of Theorem 8. Since $\text{Ces}_1[0, 1] = L^1(\ln 1/t)$ (with equality of norms) and $L^1(\ln 1/t)$ is isometric to L^1 , then in the case $p = 1$ the result follows from the Szlenk theorem [55]. Therefore, we will consider the case when $1 < p < \infty$. Taking into account Proposition 7 it is enough to prove the following: if $\{x_n\} \subset \text{Ces}_p$, $x_n \xrightarrow{w} 0$ and either

(a) $\{x_n\}$ consists of elements with equicontinuous norms

or

(b) $\text{supp } x_n \subset I_n$, where $I_n \rightarrow 1$

or

(c) $\text{supp } x_n \subset I_n$, where $I_n \rightarrow 0$,

then there is a subsequence $\{x'_n\} \subset \{x_n\}$ such that

$$(41) \quad \frac{1}{m} \left\| \sum_{k=1}^m x'_k \right\|_{C(p)} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Case (a). We will use the following remark from Szlenk paper [55, Remark 1]: a sequence $\{x_n\} \subset X, x_n \xrightarrow{w} 0$ in X (X is a Banach space) contains a subsequence $\{x'_n\}$ such that $\frac{1}{m} \|\sum_{k=1}^m x'_k\|_X \rightarrow 0$ as $m \rightarrow \infty$ if and only if it contains a subsequence $\{x_{n_k}\}$ such that

$$(42) \quad \lim_{m \rightarrow \infty} \sup_{k_1 < \dots < k_m} \left\| \frac{1}{m} \sum_{i=1}^m x_{n_{k_i}} \right\|_X = 0.$$

Let $\{x_n\} \subset \text{Ces}_p, x_n \xrightarrow{w} 0$ and $\varepsilon > 0$. At first, setting

$$A_{n,m} = \{t \in [0, 1]: |x_n(t)| \geq m\}, \quad m, n = 1, 2, \dots,$$

we prove that

$$(43) \quad \lim_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} m(A_{n,m}) = 0.$$

We may assume that $\|x_n\|_{C(p)} = 1 (n = 1, 2, \dots)$. Therefore,

$$\begin{aligned} 1 = \|x_n\|_{C(p)} &\geq \|x_n\|_{C(1)} = \|x_n\|_{L^1(\ln 1/t)} = \int_0^1 |x_n(t)| \ln \frac{1}{t} dt \\ &\geq \int_{A_{n,m}} |x_n(t)| \ln \frac{1}{t} dt \geq m \int_{A_{n,m}} \ln \frac{1}{t} dt, \end{aligned}$$

i.e.,

$$(44) \quad \int_{A_{n,m}} \ln \frac{1}{t} dt \leq \frac{1}{m} \quad \text{for all } n, m \in \mathbb{N}.$$

Assume that (43) does not hold, that is, there exists a $\delta > 0$ such that for every $m \in \mathbb{N}$ there is $n_m \in \mathbb{N}$ such that $m(A_{n_m,m}) > \delta$. Clearly, we may assume that $n_m \rightarrow \infty$ as $m \rightarrow \infty$. Since

$$m\left(A_{n_m,m} \cap \left[0, 1 - \frac{\delta}{2}\right]\right) > \frac{\delta}{2},$$

then we have that, for any $m \in \mathbb{N}$,

$$\begin{aligned} \int_{A_{n_m,m}} \ln \frac{1}{t} dt &\geq \int_{A_{n_m,m} \cap [0, 1 - \frac{\delta}{2}]} \ln \frac{1}{t} dt \\ &\geq \ln \frac{2}{2-\delta} m\left(A_{n_m,m} \cap \left[0, 1 - \frac{\delta}{2}\right]\right) > \frac{\delta}{2} \ln \frac{2}{2-\delta}. \end{aligned}$$

The last inequality contradicts (44) and, therefore, (43) is proved.

We recall that $\{x_n\}$ consists of functions having equicontinuous norms. Hence, by (43), for some m_0 and all $n \in \mathbb{N}$,

$$(45) \quad \|x_n \chi_{A_{n,m_0}}\|_{C(p)} < \frac{\varepsilon}{3}.$$

Denote $y_n = x_n \chi_{A_{n,m_0}}$ ($n = 1, 2, \dots$). Then $|x_n(t) - y_n(t)| \leq m_0$ for $t \in [0, 1]$, so that, in particular, $x_n - y_n \in L^p$ and $\|x_n - y_n\|_{L^p} \leq m_0$. Since L^p is a reflexive space for $1 < p < \infty$ and since L^p has the Banach–Saks property (cf. [7, Chapter XII, Theorem 2]), we may choose an increasing sequence of natural numbers $\{n_k\}_{k=1}^\infty$ such that $x_{n_k} - y_{n_k} \xrightarrow{w} v$ in L^p , where $v \in L^p$, and (see (42))

$$(46) \quad \lim_{m \rightarrow \infty} \sup_{k_1 < \dots < k_m} \left\| \frac{1}{m} \sum_{i=1}^m (x_{n_{k_i}} - y_{n_{k_i}}) - v \right\|_{L^p} = 0.$$

Using the imbedding $L^p \subset \text{Ces}_p$ (see Theorem 1(c)) we obtain that $x_{n_k} - y_{n_k} \xrightarrow{w} v$ in Ces_p . Therefore, since $x_{n_k} \xrightarrow{w} 0$ in Ces_p , we get that $y_{n_k} \xrightarrow{w} -v$ in Ces_p . Moreover, from (45) it follows that $\|y_{n_k}\|_{C(p)} < \frac{\varepsilon}{3}$ so that $\|v\|_{C(p)} \leq \frac{\varepsilon}{3}$. At last, by Theorem 1(c) and (46), for large enough $m \in \mathbb{N}$,

$$\begin{aligned} & \sup_{k_1 < \dots < k_m} \left\| \frac{1}{m} \sum_{i=1}^m (x_{n_{k_i}} - y_{n_{k_i}}) - v \right\|_{C(p)} \\ & \leq p' \sup_{k_1 < \dots < k_m} \left\| \frac{1}{m} \sum_{i=1}^m (x_{n_{k_i}} - y_{n_{k_i}}) - v \right\|_{L^p} \leq \frac{\varepsilon}{3}. \end{aligned}$$

The last inequalities give us that

$$\begin{aligned} & \sup_{k_1 < \dots < k_m} \left\| \frac{1}{m} \sum_{i=1}^m x_{n_{k_i}} \right\|_{C(p)} \\ & \leq \sup_{k_1 < \dots < k_m} \left(\left\| \frac{1}{m} \sum_{i=1}^m (x_{n_{k_i}} - y_{n_{k_i}}) - v \right\|_{C(p)} \right. \\ & \quad \left. + \frac{1}{m} \left\| \sum_{i=1}^m y_{n_{k_i}} \right\|_{C(p)} + \|v\|_{C(p)} \right) \leq \varepsilon, \end{aligned}$$

if m is large enough. Thus, $\{x_{n_k}\}$ satisfies condition (42) and in this case everything is proved.

Case (b). Let $\{x_n\} \subset \text{Ces}_p$, $x_n \xrightarrow{w} 0$, $\text{supp } x_n \subset I_n = [a_n, b_n]$ ($n = 1, 2, \dots$), $I_1 < I_2 < \dots$ and $a_n \rightarrow 1^-$. We may suppose that $x_n \geq 0$, $\|x_n\|_{C(p)} = 1$ and $a_1 \geq 1/2$.

Since $p > 1$, it is enough to show that $\{x_n\}$ contains a subsequence (for simplicity, it will be denoted also by $\{x_n\}$) such that

$$(47) \quad \left\| \sum_{k=1}^n x_k \right\|_{C(p)} \leq C n^{1/p},$$

where $C > 0$ is independent of $n \in \mathbb{N}$. We will choose x_n inductively. Suppose that $m \geq 2$ and x_1, x_2, \dots, x_{m-1} are already chosen. Then $a_1 < b_1 \leq \dots \leq a_{m-1} < b_{m-1} < 1$ are fixed and, since $a_n \rightarrow 1^-$, we may take a_m so that

$$(48) \quad 1 - a_m \leq (1 - b_{m-1}) \cdot 2^{-p}.$$

Then for x_m we take the function corresponding to the interval $I_m = [a_m, b_m]$ (that is, $\text{supp } x_m \subset I_m$). Let's check that inequality (47) holds. For all $n \in \mathbb{N}$ and $t \in (0, 1]$ we have that

$$\begin{aligned} \frac{1}{t} \int_0^t \left| \sum_{k=1}^n x_k(s) \right| ds &= \frac{1}{t} \sum_{m=1}^n \left(\sum_{i=1}^{m-1} \int_{a_i}^{b_i} x_i(s) ds + \int_{a_m}^t x_m(s) ds \right) \chi_{[a_m, b_m]}(t) \\ &\quad + \frac{1}{t} \sum_{m=1}^n \sum_{i=1}^m \int_{a_i}^{b_i} x_i(s) ds \chi_{[b_m, a_{m+1}]}(t) \\ &= S_1(t) + S_2(t) + S_3(t), \end{aligned}$$

where $a_{n+1} = 1$ and

$$\begin{aligned} S_1(t) &= \frac{1}{t} \sum_{m=2}^n \sum_{i=1}^{m-1} \int_{a_i}^{b_i} x_i(s) ds \chi_{[a_m, b_m]}(t), \\ S_2(t) &= \frac{1}{t} \sum_{m=1}^n \sum_{i=1}^m \int_{a_i}^{b_i} x_i(s) ds \chi_{[b_m, a_{m+1}]}(t), \\ S_3(t) &= \frac{1}{t} \sum_{m=1}^n \int_{a_m}^t x_m(s) ds \chi_{[a_m, b_m]}(t). \end{aligned}$$

Since, by Theorem 3, $(\text{Ces}_p[0, 1])^* = (\text{Ces}_p[0, 1])' = U(p')$ it follows that, for all $i \in \mathbb{N}$,

$$\begin{aligned} \int_{a_i}^{b_i} x_i(s) ds &\leq A \|x_i\|_{C(p)} \cdot \|\chi_{[a_i, b_i]}\|_{U(p')} = A \left(\int_0^{b_i} \frac{dt}{(1-t)^{p'}} \right)^{1/p'} \\ &= A(p-1)^{1/p'} \left[\frac{1}{(1-b_i)^{p'-1}} - 1 \right]^{1/p'} \leq \frac{B}{(1-b_i)^{1/p}}, \end{aligned}$$

where $B > 0$ depends only on p . Moreover, by (48), for every $i = 1, 2, \dots, m-1$,

$$\left(\frac{1-a_m}{1-b_i} \right)^{1/p} \leq \prod_{j=i}^{m-1} \left(\frac{1-a_{j+1}}{1-b_j} \right)^{1/p} \leq 2^{i-m}.$$

Therefore,

$$\begin{aligned}
\|S_1\|_p^p &= \sum_{m=2}^n \left(\sum_{i=1}^{m-1} \int_{a_i}^{b_i} x_i(s) ds \right)^p \int_{a_m}^{b_m} \frac{dt}{t^p} \\
&\leq B^p \sum_{m=2}^n \left(\sum_{i=1}^{m-1} (1-b_i)^{-1/p} \right)^p \frac{b_m^{p-1} - a_m^{p-1}}{(p-1)a_m^{p-1}b_m^{p-1}} \\
&\leq C_1^p \sum_{m=2}^n \left(\sum_{i=1}^{m-1} (1-b_i)^{-1/p} \right)^p (1-a_m) \\
&\leq C_1^p \sum_{m=2}^n \left(\sum_{i=1}^{m-1} 2^{i-m} \right)^p \leq C_1^p n,
\end{aligned}$$

so that $\|S_1\|_p \leq C_1 n^{1/p}$, where $C_1 > 0$ depends only on p . Similarly,

$$\begin{aligned}
\|S_2\|_p^p &\leq B^p \sum_{m=1}^n \left(\sum_{i=1}^m (1-b_i)^{-1/p} \right)^p \frac{a_{m+1}^{p-1} - b_m^{p-1}}{(p-1)a_{m+1}^{p-1}b_m^{p-1}} \\
&\leq C_2^p \sum_{m=1}^n \left(\sum_{i=1}^m \left(\frac{1-b_m}{1-b_i} \right)^{1/p} \right)^p \\
&\leq C_2^p \sum_{m=1}^n \left(1 + \sum_{i=1}^{m-1} 2^{i-m} \right)^p \leq (2C_2)^p n,
\end{aligned}$$

which implies that $\|S_2\|_p \leq 2C_2 n^{1/p}$, where $C_2 > 0$ depends only on p . Finally, it is easy to see that

$$\|S_3\|_p \leq \left(\sum_{m=1}^n \|x_m\|_{C(p)}^p \right)^{1/p} = n^{1/p}.$$

Thus, combining the estimates of S_1 , S_2 and S_3 we get

$$\left\| \sum_{k=1}^n x_k \right\|_{C(p)} \leq \sum_{k=1}^3 \|S_k\|_p \leq (1 + C_1 + 2C_2) \cdot n^{1/p},$$

where $C := 1 + C_1 + 2C_2$ is independent of $n \in \mathbb{N}$, that is, inequality (47) is proved.

Case (c). Let $\{x_n\} \subset \text{Ces}_p$, $x_n \xrightarrow{w} 0$, $\text{supp } x_n \subset I_n = [a_n, b_n]$ ($n = 1, 2, \dots$), $I_1 > I_2 > \dots$ and $b_n \rightarrow 0^+$. Again we may assume that $x_n \geq 0$, $\|x_n\|_{C(p)} = 1$ and $b_1 \leq 1/2$. As in the case (b) it is enough to prove inequality (47) for some subsequence of $\{x_n\}$ (it will be denoted also by $\{x_n\}$), which will be chosen inductively.

Suppose that $m \geq 2$ and the functions x_1, x_2, \dots, x_{m-1} are chosen. Then $b_1 > a_1 \geq \dots \geq b_{m-1} > a_{m-1} > 0$ are fixed and, since $b_n \rightarrow 0^+$, we may take b_m so that

$$(49) \quad b_m \leq 2^{-p'} a_{m-1}.$$

Let us show that the corresponding subsequence $\{x_n\}$ satisfies inequality (47). For any $n \in \mathbb{N}$ and $t \in (0, 1]$ we have that

$$\begin{aligned}
 & \frac{1}{t} \int_0^t \left| \sum_{k=1}^n x_k(s) \right| ds \\
 &= \frac{1}{t} \sum_{j=1}^n \left(\sum_{i=n-j+2}^n \int_{a_i}^{b_i} x_i(s) ds + \int_{a_{n-j+1}}^t x_{n-j+1}(s) ds \right) \chi_{[a_{n-j+1}, b_{n-j+1}]}(t) \\
 & \quad + \frac{1}{t} \sum_{j=1}^n \sum_{i=n-j+1}^n \int_{a_i}^{b_i} x_i(s) ds \chi_{[b_{n-j+1}, a_{n-j}]}(t) \\
 &= T_1(t) + T_2(t) + T_3(t),
 \end{aligned}$$

where $a_0 = 1$ and

$$\begin{aligned}
 T_1(t) &= \frac{1}{t} \sum_{j=2}^n \sum_{i=n-j+2}^n \int_{a_i}^{b_i} x_i(s) ds \chi_{[a_{n-j+1}, b_{n-j+1}]}(t), \\
 T_2(t) &= \frac{1}{t} \sum_{j=1}^n \sum_{i=n-j+1}^n \int_{a_i}^{b_i} x_i(s) ds \chi_{[b_{n-j+1}, a_{n-j}]}(t), \\
 T_3(t) &= \frac{1}{t} \sum_{j=1}^n \int_{a_{n-j+1}}^t x_{n-j+1}(s) ds \chi_{[a_{n-j+1}, b_{n-j+1}]}(t).
 \end{aligned}$$

Using again the duality result, as in the proof of (b), we find that

$$\begin{aligned}
 \int_{a_i}^{b_i} x_i(s) ds &\leq A \|x_i\|_{C(p)} \cdot \|\chi_{[a_i, b_i]}\|_{U(p')} \\
 &= A(p-1)^{1/p'} \left[\frac{1 - (1 - b_i)^{p'-1}}{(1 - b_i)^{p'-1}} \right]^{1/p'} \\
 &\leq B' b_i^{1/p'} \quad (i = 1, 2, \dots),
 \end{aligned}$$

where $B' > 0$ depends only on p . Since, by (49), for any $k < i$

$$\left(\frac{b_i}{a_k} \right)^{1/p'} \leq \prod_{m=k}^i \left(\frac{b_m}{a_{m-1}} \right)^{1/p'} \leq 2^{k-i-1},$$

then

$$\begin{aligned}
\|T_1\|_p^p &= \sum_{j=2}^n \left(\sum_{i=n-j+2}^n \int_{a_i}^{b_i} x_i(s) ds \right)^p \int_{a_{n-j+1}}^{b_{n-j+1}} \frac{dt}{t^p} \\
&\leq (B')^p \sum_{j=2}^n \left(\sum_{i=n-j+2}^n b_i^{1/p'} \right)^p \frac{b_{n-j+1}^{p-1} - a_{n-j+1}^{p-1}}{(p-1)a_{n-j+1}^{p-1}b_{n-j+1}^{p-1}} \\
&\leq B_1^p \sum_{j=2}^n \left(\sum_{i=n-j+2}^n b_i^{1/p'} \right)^p a_{n-j+1}^{1-p} \\
&\leq B_1^p \sum_{j=2}^n \left(\sum_{i=n-j+2}^n 2^{n-i-j+1} \right)^p \leq B_1 n,
\end{aligned}$$

so that $\|T_1\|_p \leq B_1 n^{1/p}$, where $B_1 > 0$ depends only on p . Similarly,

$$\begin{aligned}
\|T_2\|_p^p &\leq (B')^p \sum_{j=1}^n \left(\sum_{i=n-j+1}^n b_i^{1/p'} \right)^p \frac{a_{n-j}^{p-1} - b_{n-j+1}^{p-1}}{(p-1)a_{n-j}^{p-1}b_{n-j+1}^{p-1}} \\
&\leq B_2^p \sum_{j=1}^n \left(\sum_{i=n-j+1}^n \left(\frac{b_i}{b_{n-j+1}} \right)^{1/p'} \right)^p \\
&\leq B_2^p \sum_{j=1}^n \left(1 + \sum_{i=n-j+2}^n 2^{n-i-j+1} \right)^p \leq (2B_2)^p n.
\end{aligned}$$

Hence, $\|T_2\|_p \leq 2B_2 n^{1/p}$, where $B_2 > 0$ depends only on p . Moreover, it is clear that

$$\|T_3\|_p \leq \left(\sum_{j=1}^n \|x_j\|_{C(p)}^p \right)^{1/p} = n^{1/p}.$$

Thus, combining the estimates of T_1 , T_2 and T_3 we get that

$$\left\| \sum_{k=1}^n x_k \right\|_{C(p)} \leq \sum_{k=1}^3 \|T_k\|_p \leq (1 + B_1 + 2B_2) \cdot n^{1/p},$$

where $B := 1 + B_1 + 2B_2$ is independent of $n \in \mathbb{N}$. Since all cases (a)–(c) are examined, the theorem is proved. \square

8. THE CESÀRO FUNCTION SPACES $\text{Ces}_p[0, \infty)$ AND $\text{Ces}_p[0, 1]$ ARE ISOMORPHIC FOR $1 < p \leq \infty$

The main result in this Section is a construction of an isomorphism between the Cesàro function spaces $\text{Ces}_p[0, \infty)$ and $\text{Ces}_p[0, 1]$ for $1 < p \leq \infty$.

Theorem 9. If $1 < p \leq \infty$, then the Cesàro function spaces $\text{Ces}_p[0, 1]$ and $\text{Ces}_p[0, \infty)$ are isomorphic.

Proof. The proof will go in two parts. Let $1 < p < \infty$. Sy, Zhang and Lee proved in [54] that the norm in $\text{Ces}_p[0, \infty)$ is equivalent to the functional

$$(50) \quad \|f\|_0 = \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n s_k(f) \right)^p + \sum_{m=1}^{\infty} \left(m \sum_{k=m}^{\infty} t_k(f) \right)^p m^{-2} \right]^{1/p},$$

where

$$s_k(f) = \int_k^{k+1} |f(s)| ds \quad \text{and} \quad t_k(f) = \int_{\frac{1}{k+1}}^{\frac{1}{k}} |f(s)| ds, \quad k = 1, 2, \dots$$

Let's prove the analogous assertion for the space $\text{Ces}_p[0, 1]$. At first, if $\frac{1}{m+1} \leq x \leq \frac{1}{m}$, $m = 1, 2, \dots$, then

$$\frac{m+1}{2} \int_0^{1/(m+1)} |f| \leq \frac{1}{x} \int_0^x |f| \leq (m+1) \int_0^x |f| \leq 2m \int_0^{1/m} |f|.$$

Therefore,

$$\begin{aligned} & 2^{-p} \sum_{m=1}^{\infty} \left((m+1) \int_0^{1/(m+1)} |f| \right)^p \left(\frac{1}{m} - \frac{1}{m+1} \right) \\ & \leq \sum_{m=1}^{\infty} \int_{1/(m+1)}^{1/m} \left(\frac{1}{x} \int_0^x |f| \right)^p dx \\ & = \int_0^1 \left(\frac{1}{x} \int_0^x |f| \right)^p dx \end{aligned}$$

and

$$\begin{aligned} \int_0^{1/2} \left(\frac{1}{x} \int_0^x |f| \right)^p dx &= \sum_{m=2}^{\infty} \int_{1/(m+1)}^{1/m} \left(\frac{1}{x} \int_0^x |f| \right)^p dx \\ &\leq 2^p \sum_{m=2}^{\infty} \left(m \int_0^{1/m} |f| \right)^p \left(\frac{1}{m} - \frac{1}{m+1} \right). \end{aligned}$$

The first of these inequalities implies that

$$\begin{aligned}
 & 2^{-p} \sum_{m=2}^{\infty} \left(m \int_0^{1/m} |f| \right)^p \left(\frac{1}{m} - \frac{1}{m+1} \right) \\
 &= 2^{-p} \sum_{m=1}^{\infty} \left((m+1) \int_0^{1/(m+1)} |f| \right)^p \left(\frac{1}{m+1} - \frac{1}{m+2} \right) \\
 &\leq \int_0^1 \left(\frac{1}{x} \int_0^x |f| \right)^p dx = \|f\|_{C(p)}^p,
 \end{aligned}$$

and the second one yields that

$$(51) \quad 2^{-p} \int_0^{1/2} \left(\frac{1}{x} \int_0^x |f| \right)^p dx \leq \sum_{m=2}^{\infty} \left(m \int_0^{1/m} |f| \right)^p \left(\frac{1}{m} - \frac{1}{m+1} \right) \leq 2^p \|f\|_{C(p)}^p.$$

Denote

$$\alpha_n = \frac{1}{2}(2 - n^{1-p}), \quad n = 1, 2, \dots$$

It is easy to check that $\frac{1}{2} = \alpha_1 \leq \alpha_n < \alpha_{n+1} \leq 2\alpha_n$, $n = 1, 2, \dots$, and $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$. Thus, if $\alpha_n \leq x \leq \alpha_{n+1}$, then

$$\begin{aligned}
 \frac{1}{2\alpha_n} \int_0^{\alpha_n} |f| &\leq \frac{1}{\alpha_{n+1}} \int_0^{\alpha_n} |f| \leq \frac{1}{x} \int_0^x |f| \\
 &\leq \frac{1}{\alpha_n} \int_0^{\alpha_{n+1}} |f| \leq \frac{2}{\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f|,
 \end{aligned}$$

which implies that

$$\begin{aligned}
 (52) \quad & 2^{-p} \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_n} \int_0^{\alpha_n} |f| \right)^p (\alpha_{n+1} - \alpha_n) \\
 &\leq \sum_{n=1}^{\infty} \int_{\alpha_n}^{\alpha_{n+1}} \left(\frac{1}{x} \int_0^x |f| \right)^p dx \\
 &= \int_{1/2}^1 \left(\frac{1}{x} \int_0^x |f| \right)^p dx
 \end{aligned}$$

and

$$\begin{aligned}
 (53) \quad \int_{1/2}^1 \left(\frac{1}{x} \int_0^x |f| \right)^p dx &= \sum_{n=1}^{\infty} \int_{\alpha_n}^{\alpha_{n+1}} \left(\frac{1}{x} \int_0^x |f| \right)^p dx \\
 &\leq 2^p \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f| \right)^p (\alpha_{n+1} - \alpha_n).
 \end{aligned}$$

Moreover, since

$$\alpha_{n+1} - \alpha_n = \frac{1}{2} \left(\frac{1}{n^{p-1}} - \frac{1}{(n+1)^{p-1}} \right) = \frac{p-1}{2} \int_n^{n+1} \frac{1}{t^p} dt$$

and $\frac{1}{n^p} \geq \int_n^{n+1} \frac{1}{t^p} dt \geq \frac{1}{(n+1)^p} \geq \frac{1}{(2n)^p} = 2^{-p} \frac{1}{n^p}$, it follows that

$$(54) \quad \frac{p-1}{2^{p+1}n^p} \leq \alpha_{n+1} - \alpha_n \leq \frac{p-1}{2n^p}, \quad n = 1, 2, \dots,$$

and we conclude that

$$\begin{aligned}
 \alpha_{n+1} - \alpha_n &\leq \frac{p-1}{2n^p} \leq 4^p \frac{p-1}{2^{p+1}(n+1)^p} \\
 &\leq 4^p (\alpha_{n+2} - \alpha_{n+1}), \quad n = 1, 2, \dots
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 &\sum_{n=1}^{\infty} \left(\frac{1}{\alpha_n} \int_0^{\alpha_n} |f| \right)^p (\alpha_{n+1} - \alpha_n) \\
 &= \left(\frac{1}{\alpha_1} \int_0^{\alpha_1} |f| \right)^p (\alpha_2 - \alpha_1) + \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f| \right)^p (\alpha_{n+2} - \alpha_{n+1}) \\
 &\geq 4^{-p} \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f| \right)^p (\alpha_{n+1} - \alpha_n).
 \end{aligned}$$

By combining the last inequality with (52) we obtain that

$$(55) \quad \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f| \right)^p (\alpha_{n+1} - \alpha_n) \leq 8^p \int_{1/2}^1 \left(\frac{1}{x} \int_0^x |f| \right)^p dx.$$

From (51), (53) and (55) it follows that

$$\begin{aligned}\|f\|_{C(p)}^p &\leq 2^p \sum_{m=2}^{\infty} \left(m \int_0^{1/m} |f| \right)^p \left(\frac{1}{m} - \frac{1}{m+1} \right) \\ &\quad + 2^p \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f| \right)^p (\alpha_{n+1} - \alpha_n)\end{aligned}$$

and

$$\begin{aligned}&\sum_{m=2}^{\infty} \left(m \int_0^{1/m} |f| \right)^p \left(\frac{1}{m} - \frac{1}{m+1} \right) + \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f| \right)^p (\alpha_{n+1} - \alpha_n) \\ &\leq 2^p \int_0^1 \left(\frac{1}{x} \int_0^x |f| \right)^p dx + 8^p \int_{1/2}^1 \left(\frac{1}{x} \int_0^x |f| \right)^p dx \\ &\leq 2^p (4^p + 1) \|f\|_{C(p)}^p.\end{aligned}$$

Thus, taking into account (54), we obtain that

$$\begin{aligned}\|f\|_{C(p)} &\approx \left[\sum_{n=1}^{\infty} \left(\frac{1}{n\alpha_{n+1}} \int_0^{\alpha_{n+1}} |f(t)| dt \right)^p \right. \\ (56) \quad &\quad \left. + \sum_{m=2}^{\infty} \left(m \int_0^{1/m} |f(t)| dt \right)^p m^{-2} \right]^{1/p}.\end{aligned}$$

Note that

$$\begin{aligned}\int_0^{1/m} |f(t)| dt &= \sum_{k=m}^{\infty} t_k(f), \\ \text{where } t_k(f) &= \int_{1/(k+1)}^{1/k} |f(t)| dt,\end{aligned}$$

and

$$\begin{aligned}\int_0^{\alpha_{n+1}} |f(t)| dt &= \int_0^{1/2} |f(t)| dt + \sum_{k=1}^n b_k(f), \\ \text{where } b_k(f) &= \int_{\alpha_k}^{\alpha_{k+1}} |f(t)| dt.\end{aligned}$$

Since $\alpha_n \geq \frac{1}{2}$ ($n = 1, 2, \dots$) it follows that the first sum in (56) does not exceed

$$\begin{aligned} & 2^p \sum_{n=1}^{\infty} \left(\int_0^{1/2} |f(t)| dt + \sum_{k=1}^n b_k(f) \right)^p n^{-p} \\ & \leq 2^{2p} \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n b_k(f) \right)^p + \left(\int_0^{1/2} |f(t)| dt \right)^p \sum_{n=1}^{\infty} n^{-p} \right]. \end{aligned}$$

Because $p > 1$ and the second sum on the right-hand side of (56) contains $(\int_0^{1/2} |f(t)| dt)^p$, then

$$(57) \quad \|f\|_{C(p)} \approx \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n b_k(f) \right)^p + \sum_{m=2}^{\infty} \left(m \sum_{k=m}^{\infty} t_k(f) \right)^p m^{-2} \right]^{1/p}.$$

Denote by k_n and l_m one-to-one affine mappings such that

$$\begin{aligned} k_n &: [n, n+1] \rightarrow [\alpha_n, \alpha_{n+1}], \\ l_m &: \left[\frac{1}{m+1}, \frac{1}{m} \right] \rightarrow \left[\frac{1}{m+2}, \frac{1}{m+1} \right] \end{aligned} \quad (n, m = 1, 2, \dots)$$

and define the linear operator T for $f \in \text{Ces}_p[0, 1]$ by

$$\begin{aligned} Tf(x) &= \sum_{n=1}^{\infty} (\alpha_{n+1} - \alpha_n) f(k_n(x)) \chi_{[n, n+1]}(x) \\ &\quad + \sum_{m=1}^{\infty} f(l_m(x)) \chi_{[\frac{1}{m+1}, \frac{1}{m}]}(x). \end{aligned}$$

Since

$$\int_n^{n+1} |f(k_n(x))| dx = \frac{1}{\alpha_{n+1} - \alpha_n} \int_{\alpha_n}^{\alpha_{n+1}} |f(t)| dt$$

and

$$\int_{1/(m+1)}^{1/m} |f(l_m(x))| dx = \frac{m+2}{m} \int_{1/(m+2)}^{1/(m+1)} |f(t)| dt$$

for $n, m = 1, 2, \dots$, then the equivalences (50) and (57) show that

$$\begin{aligned} \|Tf\|_{C_p[0, \infty)} &\approx \|Tf\|_0 \\ &= \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n s_k(Tf) \right)^p + \sum_{m=1}^{\infty} \left(m \sum_{k=m}^{\infty} t_k(Tf) \right)^p m^{-2} \right]^{1/p} \end{aligned}$$

$$\begin{aligned} &\approx \left[\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n b_k(f) \right)^p + \sum_{m=2}^{\infty} \left(m \sum_{k=m}^{\infty} t_k(f) \right)^p m^{-2} \right]^{1/p} \\ &\approx \|f\|_{C_p[0,1]}. \end{aligned}$$

Therefore, $T : \text{Ces}_p[0, 1] \rightarrow \text{Ces}_p[0, \infty)$ is an isomorphism and the proof for $1 < p < \infty$ is complete.

If $p = \infty$ the construction of isomorphism will be different and the proof is even working for the p -convexifications, that is, if $1 \leq p < \infty$, then the spaces $\text{Ces}_{\infty}^{(p)}[0, 1]$ and $\text{Ces}_{\infty}^{(p)}[0, \infty)$ are isomorphic. In particular, $\text{Ces}_{\infty}[0, 1]$ and $\text{Ces}_{\infty}[0, \infty)$ are isomorphic.

It is easy to check that

$$(58) \quad \|f\|_{C(\infty)^{(p)}[0, \infty)} \approx \sup_{k \in \mathbb{Z}} \left(2^{-k+1} \int_{\{2^{k-1} < t \leq 2^k\}} |f(t)|^p dt \right)^{1/p}$$

and

$$(59) \quad \|f\|_{C(\infty)^{(p)}[0, 1]} \approx \sup_{k=0, -1, -2, \dots} \left(2^{-k+1} \int_{\{2^{k-1} < t \leq 2^k\}} |f(t)|^p dt \right)^{1/p}.$$

Moreover, for every $k \in \mathbb{Z}$,

$$(60) \quad 2^{-k+1} \int_{2^{k-1}}^{2^k} |f(t)|^p dt = \int_0^1 |f(2^{k-1}(t+1))|^p dt.$$

Define the linear transforms

$$T_1 : \text{Ces}_{\infty}[0, \infty) \rightarrow l^{\infty} \left(\sum_{k=-\infty}^{\infty} \oplus L^p[0, 1] \right), \quad T_1 f = (f(2^{k-1}(t+1)))_{k \in \mathbb{Z}}$$

and

$$T_2 : \text{Ces}_{\infty}[0, 1] \rightarrow l^{\infty} \left(\sum_{k=0}^{-\infty} \oplus L^p[0, 1] \right), \quad T_2 f = (f(2^{k-1}(t+1)))_{k=0}^{-\infty}.$$

Formulas (58)–(60) show that T_1 and T_2 are isomorphisms. It is obvious that the spaces $l^{\infty}(\sum_{k=-\infty}^{\infty} \oplus L^p[0, 1])$ and $l^{\infty}(\sum_{k=0}^{-\infty} \oplus L^p[0, 1])$ are isomorphic. Therefore, the spaces $\text{Ces}_{\infty}^{(p)}[0, \infty)$ and $\text{Ces}_{\infty}^{(p)}[0, 1]$ are isomorphic. \square

Problem 1. Is the Cesàro function space $\text{Ces}_{\infty}(I)$ isomorphic with the Cesàro sequence space ces_{∞} ?

In Theorem 6 we proved that $\text{Ces}_p[0, 1]$ contains an isomorphic copy of l^p . Now we try to investigate when this is true for the spaces l^q .

Theorem 10.

- (a) *If $1 \leq p \leq 2$, then the space l^q is embedded isomorphically into $\text{Ces}_p[0, 1]$ if and only if $q \in [1, 2]$.*
- (b) *If $2 < p < \infty$, then the space l^q is embedded isomorphically into $\text{Ces}_p[0, 1]$ if and only if either $q \in [1, 2]$ or $q = p$.*

Proof. Firstly, $\text{Ces}_p[0, 1]$ contains a copy of $L^1[0, 1]$ (cf. [4, Lemma 1]) and in turn l^q is embedded into $L^1[0, 1]$ if $1 \leq q \leq 2$ (cf. [1, Theorem 6.4.18]). Moreover, by Theorem 6, l^p is embedded into $\text{Ces}_p[0, 1]$ for every $p \in [1, \infty)$ so we have to prove only the necessity.

In the case when $1 \leq p \leq 2$ necessity is obvious as a consequence of the fact that $\text{Ces}_p[0, 1]$ has cotype 2.

If $p > 2$ noting that $\text{Ces}_p[0, 1] \subset \text{Ces}_1[0, 1] = L^1(\ln 1/t)$ we consider two cases:

(a) Assume that the norms of the spaces $\text{Ces}_p[0, 1]$ and $L^1(\ln 1/t)$ are equivalent on a subspace $X \subset \text{Ces}_p[0, 1]$ which is isomorphic to l^q . In other words, X is a subspace of $L^1(\ln 1/t)$. Since the last space has cotype 2, then $q \leq 2$.

(b) The norms of the spaces $\text{Ces}_p[0, 1]$ and $L^1(\ln 1/t)$ are not equivalent on $X \approx l^q$. Then there is a sequence $\{x_n\} \subset X$ such that $\|x_n\|_{C(p)} = 1$ and $\|x_n\|_{L^1(\ln 1/t)} \rightarrow 0$. In particular, $x_n \rightarrow 0$ weakly in $L^1(\ln 1/t)$, i.e.,

$$\int_0^1 x_n(t)y(t) dt \rightarrow 0 \quad \text{for every } y \in L^\infty(\ln^{-1} 1/t).$$

Denote $\mathcal{F} := \bigcup_{0 < \delta < 1} L^\infty[0, \delta]$. Obviously, it yields that $\mathcal{F} \subset L^\infty(\ln^{-1} 1/t)$ and \mathcal{F} is dense in $(\text{Ces}_p[0, 1])' = U(p')$ (see Theorem 3). Therefore, $\|x_n\|_{C(p)} = 1$ and $x_n \rightarrow 0$ weakly in $\text{Ces}_p[0, 1]$. By a known result (cf. [37, Proposition 1.a.12]) there exists a subsequence $\{x'_n\} \subset \{x_n\}$ which is equivalent to a seminormalized block basis of the canonical basis of l^q and, consequently, is equivalent to the canonical basis of l^q itself (see [1, Lemma 2.1.1 and Remark 2.1.2]). Moreover, $\|x'_n\|_{C(p)} = 1$ and $x'_n \rightarrow 0$ in the Lebesgue measure m . Next, since $\text{Ces}_p[0, 1]$ is separable for $1 \leq p < \infty$, then applying the Kadec–Pełczyński procedure we may find a subsequence $\{x''_n\} \subset \{x'_n\}$ and a sequence of disjoint sets $A_n \subset [0, 1]$ such that $\|x''_n - x''_n \chi_{A_n}\|_{C(p)} \rightarrow 0$. Using a standard argument we can select a subsequence $\{x''_{n_k}\} \subset \{x''_n\}$, which is equivalent to the sequence of disjoint functions $z_k := x''_{n_k} \chi_{A_{n_k}}$. Note that $\{x''_{n_k}\}$ and $\{z_k\}$ as well are equivalent to the canonical basis of l^q . To show that either $q \in [1, 2]$ or $q = p$ we consider separately two cases:

(1) firstly, assume that there is $h \in (0, \frac{1}{2})$ such that $\text{supp } z_k \subset [h, 1 - h]$ for all $k = 1, 2, \dots$. Since $\text{Ces}_p[h, 1 - h] \simeq L^1[h, 1 - h]$ (cf. [4, Lemma 1]), then l^q will be embedded into $L^1[h, 1 - h] \simeq L^1[0, 1]$, so that $q \in [1, 2]$.

(2) otherwise, there is a subsequence $\{z'_k\} \subset \{z_k\}$ such that $\text{supp } z_k \subset I_k$ for some intervals I_k satisfying either $I_k \rightarrow 0$ or $I_k \rightarrow 1$. Then, using the same arguments as in the proof of Theorem 8, we may select a subsequence $\{z''_k\} \subset \{z'_k\}$ such that

$$\left\| \sum_{k=1}^m z''_k \right\|_{C(p)} \leq C m^{1/p},$$

where the constant $C > 0$ does not depend on $m = 1, 2, \dots$. Since $[z''_k] \simeq l^q$, then we have $q \geq p$. On the other hand, $q \leq p$ because $\text{Ces}_p[0, 1]$ has cotype p , thus $q = p$ and the proof is complete. \square

Let us remind that $L^p[0, 1]$ contains an isomorphic copy of l^q if and only if $q \in [p, 2]$ for the case $1 \leq p \leq 2$ and in the case when $p > 2$ this can be when either $q = p$ or $q = 2$. We can see then the difference between $L^p[0, 1]$ and $\text{Ces}_p[0, 1]$ spaces. In particular, if $1 < p < \infty$, then $\text{Ces}_p[0, 1]$ contains an isomorphic copy of l^1 but not $L^p[0, 1]$.

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